INTERMEDIATE ALGEBRA FOR COLLEGES

BY *WILLIAM L. HART*

PROFESSOR OF MATHEMATICS
UNIVERSITY OF MINNESOTA

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PREFACE

THIS BOOK offers a collegiate substitute for third semester high school algebra. The text was designed for a college student who will study it either (1) as a preliminary to taking college algebra, or (2) as terminal work in algebra which is intended as a prerequisite for elementary courses in various fields of natural or social science, or in business administration. A suitable selection of content from the book would provide a satisfactory algebraic foundation for a first course in trigonometry or in the mathematics of investment. In the case of a student of the assumed preparation, the text provides sufficient material for a substantial course utilizing from 40 to 60 class hours.

The plan of the text was based on the assumption that the typical student involved is of a mature age but studied his elementary algebra so long ago that practically all fundamentals must be taught as if they were relatively new material for him. Hence, the early chapters of the book present a mature but frankly elementary treatment of the foundations of algebraic technique with a generous amount of discussion and problem material. Also, appropriate refresher work on arithmetic is provided incidentally in the algebraic problems and explicitly in an early optional chapter devoted to computation. The tempo of the discussion in the text is gradually increased until, in the later chapters, distinctly collegiate speed is attained so that the student will find it easy to make the transition into a substantial second course devoted to college algebra. The text makes no attempt to present material which custom dictates as primarily within the sphere of college algebra, although such material frequently may enter the most substantial courses in third semester algebra at the secondary level. However, in the interest of efficiency and mathematical simplicity, the terminology and general viewpoint of the text is distinctly collegiate. Emphasis is laid on the logical sequence of topics, accuracy of definitions, and the completeness of proofs.

SPECIAL FEATURES

Adult nature of the presentation. The discussion in the text is couched at a level suitable to the maturity of college students. Hence, the available space and assumed class time are utilized mainly to explain and illustrate the mathematical principles involved and only the necessary minimum attention is devoted to artificial motivation of the type which might properly be expanded for younger students.

Terminology. Particular emphasis is given to the language concerning variables, functions, equations, and the most elementary aspects of analytic geometry because of the importance of this vocabulary in fields of application which the students will enter in college. The technical vocabulary of the algebraic content is limited by excluding terms which are of small or doubtful utility.

Illustrative material. Extensive use is made of illustrative examples to introduce new theory, to recall previous knowledge, and to furnish models for the student's solutions of problems.

Emphasis on development of skill in computation. The viewpoint is adopted that the student needs refresher training in the operations of arithmetic, as well as new mature appreciation of various features of computation. Hence, work with fractions and decimals is introduced quickly, and substantial early sections are devoted to a discussion of approximate computation. Also, the exercises and applications continue to demand computing skill throughout the text, and the chapter on logarithmic computation is made very complete.

Supplementary content. A small amount of material not essential in the typical course is segregated into obviously independent sections labeled with a black star, ★. Also, the teacher will understand that several of the later chapters are optional and that their omission in whole or in part will not interfere with the continuity of other chapters. The book was planned to eliminate the necessity for frequent omissions by the teacher.

The extent and grading of the exercises. The problem material is so abundant that, in many exercises, either the odd-numbered or the even-numbered examples alone will be found sufficient for the student's outside assignments, and the balance may be reserved for work in the classroom. In each exercise, the problems are arranged

approximately in order of increasing difficulty. Examples stated in words are emphasized at the appropriate places to avoid the development of an inarticulate form of algebraic skill. However, the text does not spend valuable time in specialized training to develop problem solving skills devoted to artificial or unimportant types of problems.

Answers. The answers to odd-numbered problems are provided in the text, and answers for even-numbered problems are furnished free in a separate pamphlet at the instructor's request.

Flexibility. The grading of the exercises, various features in the arrangement of theoretical discussions, and the location of certain chapters are designed to aid the teacher in adapting the text to the specific needs of his class.

Composition and appearance. The absence of excessively small type, the generous spacing on the pages, and the special care taken in the arrangement of the content into pages create a favorable setting for the use of the book by both the teacher and the student.

University of Minnesota WILLIAM L. HART

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CHAPTER *1*

THE FUNDAMENTAL OPERATIONS

1. Explicit and literal numbers

In algebra, not only do we employ explicit numbers like 2, -5, 0, etc., but, as a characteristic feature of the subject, we also use letters or other symbols to represent numbers with variable or undesignated values. For contrast with explicit numbers, we agree that number symbols such as a, b, x, and y will be called *literal* numbers. In this book, as a rule, any single letter introduced without a qualifying description will represent a number.

2. Signed Numbers *

The numbers used in the elementary stages of algebra are called **real numbers.** They are classed as *positive*, *negative*, or *zero*, 0, which is considered neither positive nor negative. The word *real* is used with reference to these numbers in order to permit contrast with a type of number called *imaginary*, which will be introduced at a later stage.

ILLUSTRATION 1. 17, $-\frac{5}{3}$, and 0 are real numbers.

In arithmetic, the numbers employed consist of zero, the integers or whole numbers 1, 2, 3, $\cdots$, and other unsigned numbers which we express by means of fractions or the decimal notation. These numbers, except for zero, will hereafter be called *positive* numbers. When we choose, we shall think of each positive number as having a plus sign, $+$, attached at the left.

ILLUSTRATION 2. The positive number 7 may be written $+7$ for emphasis.

* For a logical foundation for algebra, see pages 1–78 in *College Algebra*, by H. B. FINE; GINN AND COMPANY, publishers.

In a later section, we shall formally define the negative numbers, which will be described as the "*negatives*" of the positive numbers.

ILLUSTRATION 3. Corresponding to $+6$ we shall introduce the negative number -6.

Positive and negative numbers may be contrasted concretely in assigning values to quantities which are known to be of one or other of two opposite types. In such a case, we conveniently think of any positive number P and the corresponding negative number $-P$ as being *opposites*. With this in mind, we frequently refer to the signs "+" and "−" as being *opposite signs*.

ILLUSTRATION 4. In bookkeeping, if a *gain* of \$5000 is assigned the value + \$5000, a *loss* of \$3000 could be given the value − \$3000.

We shall desire the results of operations which we shall define for signed numbers to correspond with our intuitions when the numbers are interpreted concretely.

ILLUSTRATION 5. Let $t°$ indicate an *increase* of temperature when t is *positive* and a *decrease* when t is *negative*. Let time be considered *positive* in the *future* and *negative* in the *past*. Then, the following concrete statements should correspond to the indicated addition or multiplication.

A decrease of 20° *and then a rise of* 8° *creates a decrease of* 12°; *or,*

$$(-20°) + 8° = -12°.$$

A decrease of 10° *per hour in temperature for the next* 3 *hours will create a decrease of* 30°; *or,*

$$(+3) \times (-10°) = -30°.$$

If the temperature has decreased 10° *per hour for the preceding* 4 *hours, the temperature* 4 *hours ago was* 40° *higher than now; or,*

$$(-4) \times (-10°) = +40°.$$

EXERCISE 1

Under the specified condition, what meaning would be appropriate for the indicated quantity with opposite sign?

1. For − 5 miles, if + 5 miles means 5 miles *north*.

2. For + \$10, if − \$15 means \$15 *lost*.

3. For + 8°, if − 3° means a *fall* of 3° in temperature.

4. For $-14°$ latitude, if $+7°$ latitude means $7°$ *north* latitude.

5. For $-170'$ altitude, if $+20'$ altitude means $20'$ *above* sea level.

6. For $+30°$ longitude, if $-20°$ longitude means $20°$ *west* of Greenwich.

7. For $-5'$, if $+5'$ means $5'$ to the *right*.

Introduce your own agreements about signed values and express each of the following facts by adding or multiplying signed numbers.

8. A gain of \$3000 followed by a loss of \$9000 creates a loss of \$6000.

9. A fall of 40° in temperature followed by a rise of 23° creates a fall of 17°.

10. Thirty-five steps backward and then 15 steps forward bring a person to a point 20 steps backward.

11. If you have been walking forward at a rate of 25 steps per minute, then 6 minutes ago you were 150 steps back from your present position.

12. If the water level of a river is rising 4 inches per hour, then (*a*) the level will be 24 inches higher at the end of 6 hours; (*b*) the level was 36 inches lower 9 hours ago.

3. Extension of the number system

At this point, let us start with the understanding that we have at our disposal only the positive numbers and zero. Then, we shall extend this number system to include negative numbers, properly defined, and shall introduce the operations of algebra for the whole new number system. Hereafter, when we refer to *any number*, or use a *literal number* without limiting its value, we shall mean that it is any number of the *final number system* we plan to develop.

4. Algebraic operations

The **fundamental operations of algebra** are *addition*, *subtraction*, *multiplication*, and *division*. Whenever these operations are introduced, the results in applying them will be the same as in arithmetic when only positive numbers and zero are involved.

5. Multiplication

The result of multiplying two or more numbers is called their **product** and each of the given numbers is called a **factor** of their product. To indicate multiplication, we use a cross $\times$ or a high dot between the numbers, or, in the case of literal numbers, merely write them side by side without any algebraic sign between them.

We separate the factors by parentheses if the dot or cross is omitted between factors which are explicit numbers, or when a factor not at the left end of a product has a plus or a minus sign attached.

ILLUSTRATION 1. $6 \cdot 3 = 6 \times 3 = 6(3) = 18.$ We read 6×3, $6 \cdot 3$, or $6(3)$ as "*six times three.*"

ILLUSTRATION 2. $4ab$ means $4 \times a \times b$ and is read "*four a, b.*" If $a = 2$ and $b = 5$, then $4ab = 4(2)(5) = 40$.

If N is any number, we agree that

$$(+ 1) \times N = N; \qquad N \times 0 = 0. \tag{1}$$

6. Negative numbers

Let $- 1$ be a new number symbol, called "*minus* 1," to which we immediately assign the following property:

$$(- 1) \times (- 1) = + 1. \tag{1}$$

By our standard agreement about multiplication by $+ 1$,

$$(+ 1) \times (- 1) = - 1. \tag{2}$$

If P is any positive number, we introduce $- P$ as a new number symbol, called "*minus P*," to represent $(- 1) \times P$. That is,

$$- P = (- 1) \times P. \tag{3}$$

We call $- P$ a *negative number*. Our number system now consists of the positive numbers, zero, and the negative numbers.

ILLUSTRATION 1. Corresponding to $+ 6$, we have the negative number $- 6$, defined as $(- 1) \times 6$. In concrete applications, corresponding to each *positive* number, we may think of multiplication by $- 1$ as having the property of producing a number of opposite type, called *negative*.

7. Absolute value

The *absolute value* of a positive number or zero is defined as the number *itself*. The absolute value of a negative number is the given number with its sign changed from minus to plus. The absolute value of a number N is frequently represented by the symbol $| N |$.

ILLUSTRATION 1. The absolute value of $+ 5$ is $+ 5$. The absolute value of $- 5$ is also $+ 5$. We read $| - 3 |$ as "*the absolute value of* $- 3$." We have $| - 3 | = 3$ and $| 0 | = 0$.

8. Inserting signs before numbers

If we insert a plus or a minus sign before (to the *left* of) a number, this is understood to be equivalent to multiplying it by $+1$ or -1, respectively.

ILLUSTRATION 1. $+5 = (+1) \times 5 = 5.$ $\quad -16 = (-1) \times 16.$

$$+a = (+1) \times a = a. \qquad -a = (-1) \times a.$$

Hereafter, we shall act as if each explicit number or literal number expression has a sign attached, at the left. If no sign is visible, it can be assumed to be a plus sign because, for every number N, we have $N = +N$.

9. Properties of multiplication

We agree that the following postulates * are satisfied.

I. *Multiplication is commutative, or the product of two numbers is the same in whatever order they are multiplied.*

ILLUSTRATION 1. $7 \times 3 = 3 \times 7 = 21.$ $\quad ab = ba.$

ILLUSTRATION 2. $(-1) \times (+1) = (+1) \times (-1) = -1.$

II. *Multiplication is associative, or the product of three or more numbers is the same in whatever order they are grouped in multiplying.*

ILLUSTRATION 3. $5 \times 7 \times 6 = 5 \times (7 \times 6) = 7 \times (5 \times 6) = 210.$

$$abc = a(bc) = b(ac) = (ab)c.$$

We read this "*a, b, c, equals a times b, c, equals b times a, c,* etc."

ILLUSTRATION 4. The product of three or more numbers is the same in whatever order they are multiplied:

$$abc = a(bc) = (bc)a = bca = (ac)b = acb, \text{ etc.}$$

10. Computation of products

To compute a product of two numbers, find the product of their absolute values, and then

I. *give the result a* **plus** *sign if the numbers have* **like** *signs;*

II. *give the result a* **minus** *sign if the numbers have* **unlike** *signs.*

* A postulate is a property which is specified to be true as a part of the definition of the process.

The preceding facts about a product are arrived at naturally in any example by recalling the multiplication properties of -1. Statements I and II are called the **laws of signs** for multiplication.

ILLUSTRATION 1. $(-5)(-7) = +35$ because

$$(-1) \times 5 \times (-1) \times 7 = (-1)(-1)(5)(7) = (+1)(35).$$

$$(-4) \times (+7) = (-1) \times 4 \times 7 = -28.$$

In a product, the result is *positive* if an *even* number $(2, 4, 6, \cdots)$ of factors are *negative*, and the product is *negative* if an *odd* number $(1, 3, 5, \cdots)$ of factors are *negative.*

ILLUSTRATION 2. $-3(-2)(-5) = (+6)(-5) = -30.$

ILLUSTRATION 3.

$$(-1)(-1)(-1)(-1) = [(-1)(-1)][(-1)(-1)] = (+1)(+1) = 1.$$

11. Division

To divide a by b, where b is *not zero,* means to find the number x such that $a = bx$. We call a the **dividend,** b the **divisor,** and x the **quotient.** We denote the quotient by $a \div b$, or $\frac{a}{b}$, or a/b. The fraction a/b is read "*a divided by b*," or "*a over b*." In a/b, we call a the **numerator** and b the **denominator**; also, a and b are sometimes called the **terms** of the fraction. The fraction a/b, or $a \div b$, is frequently referred to as the **ratio** of a to b.

ILLUSTRATION 1. $36 \div 9 = 4$ because $4 \times 9 = 36$.

The absolute value and sign of any quotient are a consequence of the absolute value and sign of a corresponding product.

To compute a quotient of two numbers, first find the quotient of their absolute values, and then apply the laws of signs as stated for products.

ILLUSTRATION 2. $\dfrac{-40}{+10} = -4$ because $10 \times (-4) = -40$.

$$\frac{-40}{-8} = +5 \quad \text{because} \quad 5 \times (-8) = -40.$$

Note 1. Division is referred to as the *inverse* of multiplication. Thus, if 7 is first *multiplied* by 5 and if the result, 35, is then *divided* by 5, we obtain 7 unchanged. Or, division by 5 undoes the effect of multiplication by 5. Equally well, multiplication is the inverse of division.

EXERCISE 2

Read each product or quotient and give its value.

1. 7×8. **2.** $(-3) \times (-5)$. **3.** $(-2) \times (6)$.

4. $8 \times (-3)$. **5.** $(-9) \times (-4)$. **6.** $(-3) \times 5$.

7. 4×0. **8.** $(+5) \times (-3)$. **9.** $(-2) \times (+4)$

10. $(+4)(+6)$. **11.** $(-7)(-8)$. **12.** $0 \times (-3)$.

13. $(-1)(-5)$. **14.** $-(-4)$. **15.** $-(+8)$.

16. $+(-7)$. **17.** $+(+4)$. **18.** $-(+1)$.

19. $-(-3)$. **20.** $-(-1)$. **21.** $+(-9)$.

22. $(-7)(-4)(6)$. **23.** $(-2)(-7)(-3)(4)$.

24. $5(-2)(7)(-3)(-4)$. **25.** $(-5)(-3)(-4)(-2)$.

26. $-6(-4)(-3)$. **27.** $-3(5)(2)(-4)$.

28. $-(-7)(-4)$. **29.** $-4(-5)(6)(-3)$.

30. $\dfrac{+16}{+8}$. **31.** $\dfrac{-16}{8}$. **32.** $\dfrac{15}{-3}$. **33.** $\dfrac{-48}{12}$.

34. $\dfrac{-42}{-6}$. **35.** $\dfrac{-36}{-18}$. **36.** $\dfrac{-28}{+7}$. **37.** $\dfrac{+39}{-3}$.

38. $(-1)(-1)(-1)(-1)(-1)$. **39.** $(-1)(-1)(-1)(-5)$.

State the absolute value of each number.

40. 16. **41.** -52. **42.** -33. **43.** $-\frac{3}{4}$. **44.** $+14.2$.

45. Find the product of -3, -5, and -4.

46. Find the product of 5.3, -4, and $+2$.

47. Find the product of -1.8, 2, and -4.

Read each symbol and specify its value.

48. $|7|$. **49.** $|+4|$. **50.** $|-6|$. **51.** $|-31|$. **52.** $|-1.7|$.

53. Compute $4abc$ if $a = -3$, $b = -4$ and, $c = -2$.

54. Compute $3xyz$ if $x = -2$, $y = 10$, and $z = 5$.

55. Compute $2abxy$ if $a = 3$, $b = -4$, $x = -3$, and $y = 5$.

56. Compute $5hkwz$ if $h = -3$, $k = 2$, $w = -5$, and $z = -2$.

12. Addition

The result of adding two or more numbers is called their **sum.** Usually, to indicate the sum, we take each of the numbers with its attached sign, supplying a plus sign where none is written, and then *write these signed numbers in a line.* Each number, with its sign, is called a **term** of the sum. We usually omit any plus sign at the left end of a sum.

ILLUSTRATION 1. The sum of 15 and 17 is written $15 + 17$; the sum is 32, as in arithmetic.

We can state that a *plus sign* between two numbers indicates that they are to be *added,* because we could write a sum by inserting a plus sign before each term and then writing the numbers in a line. However, this might introduce unnecessary plus signs. The most useful statement is that, when numbers are written in a line, connected by their signs, *plus or minus,* this indicates that the numbers are to be *added.*

ILLUSTRATION 2. The sum of 17 and -12 is represented by $17 - 12$. Later, we will justify saying that this equals 5, the value of the expression in arithmetic. By using a needless plus sign we could have written $17 + (-12)$ for the sum.

We specify that the number -1 has the new property that the sum of -1 and $+1$ is zero. That is,

$$-1 + 1 = 0. \tag{1}$$

Also, for any number N, we agree that $N + 0 = N$.

The operation of addition satisfies the following postulates.

I. *Addition is commutative, or the sum of two numbers is the same in whatever order they are added.*

ILLUSTRATION 3. $5 + 3 = 3 + 5 = 8. \quad a + b = b + a.$

ILLUSTRATION 4. $0 = -1 + 1 = +1 - 1.$

II. *Addition is associative, or the sum of three or more numbers is the same in whatever order they are grouped in adding.*

ILLUSTRATION 5. $3 + 5 + 7 = 3 + (5 + 7) = 5 + (3 + 7) = 15.$

ILLUSTRATION 6.

$$a + b + c = a + (b + c) = a + (c + b) = a + c + b;$$
$$= c + (a + b) = c + a + b, \text{ etc.}$$

Thus, the sum of three or more numbers is the same in whatever order they are added.

Addition and multiplication satisfy the following postulate.

III. *Multiplication is distributive with respect to addition, or* *

$$a(b + c) = ab + ac.$$

ILLUSTRATION 7. $8 \times (5 + 7) = (8 \times 5) + (8 \times 7) = 40 + 56 = 96.$

13. Introduction of the negative of a number

The *negative* of a number N is defined as the result of multiplying N by -1, so that the negative of N is $-N$. The negative of a *positive* number is the corresponding *negative* number. The negative of a *negative* number is the corresponding *positive* number.

ILLUSTRATION 1. The negative of $+5$ is -5. The negative of -5 is $+5$ because

$$-(-5) = (-1) \times (-5) = +5.$$

Thus, we notice that, if one number is the negative of another, then the second number is the negative of the first.

We observe that *the sum of any number and its negative is* 0.

ILLUSTRATION 2. By Postulate III of Section 12,

$$-5 + 5 = [(-1) \times 5] + [(+1) \times 5] = 5 \times (-1 + 1) = 5 \times 0 = 0.$$
$$-a + a = [(-1) \times a] + [(+1) \times a] = a \times (-1 + 1) = a \times 0 = 0.$$

14. Subtraction

To subtract b from a will mean to find the number x which when *added* to b will yield a. This also is the definition used for subtraction in arithmetic. Hence, the result of subtracting a positive number from one which is no larger † will be *the same in algebra as in arithmetic.*

* We read $a(b + c)$ as "*a times the quantity* $b + c$."

† This is the only case of subtraction which occurs in arithmetic because negative numbers are not used in that field. Thus, $(15 - 25)$ has no meaning until negative numbers are introduced.

ILLUSTRATION 1. The result of subtracting 5 from 17 is 12 because $12 + 5 = 17$.

If x is the result of subtracting b from a, then by definition

$$a = b + x. \tag{1}$$

On recalling arithmetic, we would immediately like to write $x = a - b$, which means the *sum* of a and $-b$. To prove that $x = a - b$, we add $-b$ to a, as given in equation 1:

$$a - b = -b + a = -b + (b + x) = (-b + b) + x = x. \tag{2}$$

In (2), we proved that the result of *subtracting* b from a is obtained by *adding* $-b$ to a. Thus,

to subtract a number, add its negative. (3)

We define the **difference** of two numbers a and b as the result of subtracting the *second* number from the *first*. If x is this difference, we proved in (2) that

$$x = a - b. \tag{4}$$

Thus, we can say that the *minus sign* in (4) indicates *subtraction*, just as in arithmetic. However, it is equally important to realize that $(a - b)$ means *the sum of its two terms a and $-b$.*

ILLUSTRATION 2. The difference of 17 and 5 is $(17 - 5)$. The difference $(17 - 5)$ represents the sum of 17 and -5. Also, $(17 - 5)$ represents the result of subtracting 5 from 17, which is 12.

Note 1. In a difference $a - b$, the number b which is subtracted is called the **subtrahend**, and the number a, from which b is subtracted, is called the **minuend.** These names will not be mentioned very often.

15. Computation of a sum

The computation of a sum of two signed numbers or, as a special case, the *subtraction* of one number from another, always will lead to the use of one of the following rules.

I. *To add two numbers with like signs, add their absolute values and attach their common sign.*

II. *To add two numbers with unlike signs, subtract the smaller absolute value from the larger and prefix the sign of the number having the larger absolute value.*

ILLUSTRATION 1. $7 + 15 = 22,$ just as in arithmetic.

EXAMPLE 1. Add -5 and -17.

SOLUTION. The sum is $-5 - 17 = -(5 + 17) = -22$ by Rule I. To verify this, we use Postulate III of Section 12:

$$-5 - 17 = [(-1) \times 5] + [(-1) \times 17] = (-1) \times (5 + 17) = -22.$$

ILLUSTRATION 2. The sum of 6 and -6 is zero or, as in arithmetic, the result of subtracting 6 from 6 is zero: $6 - 6 = 0.$

ILLUSTRATION 3. The sum of 20 and -8, which is the same as the result of subtracting 8 from 20, is $20 - 8 = 12$.

ILLUSTRATION 4. The sum of -20 and 8 is, by Rule II,

$$-20 + 8 = -(20 - 8) = -12.$$

To verify this, we recall that $-20 = -12 - 8$. Hence,

$$-20 + 8 = -12 - 8 + 8 = -12 + 0 = -12.$$

Essentially, $+8$ cancels -8 of -20 and leaves -12.

EXAMPLE 2. Subtract -15 from -6.

FIRST SOLUTION. Change the sign of -15 and add:

$$result = -6 + 15 = 9.$$

SECOND SOLUTION. Write the difference of -6 and -15:

$$result = -6 - (-15) = -6 + 15 = 9.$$

CHECK. Add -15 and 9: $-15 + 9 = -6.$

16. Algebraic sums

An expression like

$$c - 3 - 5a + 7b \tag{1}$$

is referred to as a *sum*, or sometimes as an *algebraic sum* to emphasize that *minus* signs appear. Expression 1 is the sum of the *terms* c, -3, $-5a$ and $7b$. In connection with any term whose sign is minus, we could describe the effect of the term in the language of subtraction instead of addition, but frequently this is not desirable.

To compute a sum of explicit numbers, first eliminate parentheses by performing any operations indicated by the signs. Then, sometimes, it is desirable to add all positive and all negative numbers separately before combining them.

ILLUSTRATION 1. Since $+(-12) = -12$ and $-(-7) = +7$,
$$-16-(-7)+(-12)+14 = -16+7-12+14 = -28+21 = -7.$$
Or, we could compute mentally from left to right:
$$-16+7 \textit{ is } -9; \quad -12 \textit{ is } -21; \quad +14 \textit{ is } -7.$$

Note 1. It is undesirable to use *unnecessary* plus signs. Thus, in place of $+(-3)+(+8)$ we write $-3+8$.

EXAMPLE 1. Compute $(5x - 3ab)$ if $x = -3$, $a = -4$, and $b = 7$.

SOLUTION. $5x - 3ab = [5(-3)] - [3(-4)(7)] = -15 + 84 = 69.$

Comment. Notice the convenience of brackets to show that the multiplications above should be done before computing the sum.

17. Summary concerning the number zero

We have mentioned that the operation $a \div b$ is *not defined* when $b = 0$. That is, **division by zero is not allowed.** However, no exception has arisen in multiplying by zero, adding or subtracting 0, or in dividing zero by some other number. Thus, if N is any number,
$$N+0=N; \qquad N-0=N; \qquad N\times 0=0.$$
If N is not zero, then $\dfrac{0}{N} = 0$ because $0 = 0 \times N$.

ILLUSTRATION 1. $\quad 6+0=0. \qquad 3\times 0=0. \qquad \dfrac{0}{5}=0.$

Note 1. Contradictions arise when an attempt is made to define division by zero. Thus, if we were to define 5/0 to be the number x which, when multiplied by zero, will yield 5, then we would obtain $5 = 0\cdot x$, or $5 = 0$, which is contradictory.

EXERCISE 3

Compute each sum mentally.

1. $20+7$.	**2.** $-28-9$.	**3.** $-13-5$.
4. $36-4$.	**5.** $28-15$.	**6.** $12-17$.
7. $0-16$.	**8.** $6-25$.	**9.** $-13-19$.
10. $-35+6$.	**11.** $7-42$.	**12.** $0-25$.
13. $-6+6$.	**14.** $-18+3$.	**15.** $-13+0$.
16. $16.8-15.2$.	**17.** $-13.7-4.5$.	**18.** $-5+3.4$.

What is the negative of the number?

19. 7. **20.** -4. **21.** -13. **22.** 0.

Without writing the numbers in a line, (a) add the two numbers; (b) subtract the lower one from the upper one.

23. $\begin{array}{r} 45 \\ \underline{16} \end{array}$ **24.** $\begin{array}{r} 36 \\ \underline{-19} \end{array}$ **25.** $\begin{array}{r} -13 \\ \underline{-17} \end{array}$ **26.** $\begin{array}{r} -12 \\ \underline{+18} \end{array}$ **27.** $\begin{array}{r} -53 \\ \underline{+17} \end{array}$

28. $\begin{array}{r} -13 \\ \underline{-24} \end{array}$ **29.** $\begin{array}{r} 0 \\ \underline{17} \end{array}$ **30.** $\begin{array}{r} 0 \\ \underline{-15} \end{array}$ **31.** $\begin{array}{r} -4.3 \\ \underline{7.6} \end{array}$ **32.** $\begin{array}{r} -.87 \\ \underline{1.35} \end{array}$

Find the value of the expression in simple form.

33. $+(-5)$. **34.** $-(-8)$. **35.** $-(+7)$. **36.** $+(-6)$.

Compute each sum.

37. $-(-3)+7$. **38.** $+(+4)-(-8)$. **39.** $-6+(-4)$.

40. $-(-4)+(-9)$. **41.** $16-(-6)$. **42.** $-15-(+12)$.

43. $12-7-5+3$. **44.** $-16-3+5-7+6$.

45. $-10+17-8+14$. **46.** $43-25-6+8+12$.

47. $-16+14+36+8$. **48.** $3-16+17-8-9$.

49. $-3+(-5)-(-16)-(-4)$.

50. $5-7+(-3)-(+16)-3.7$.

Find (a) the sum and (b) the difference of the two numbers.

51. 16 and 12. **52.** 15 and -3. **53.** -33 and -7.

54. -14 and -5. **55.** 6 and 38. **56.** 15 and 67.

Compute the expression when the literal numbers have the given values.

57. $16+3ab$; when $a=-4$ and $b=-7$.

58. $2a-3cd$; when $a=-2$, $c=3$, and $d=-5$.

59. $2b-7+4ac$; when $a=5$, $b=7$, and $c=-3$.

60. $x-2yz-3$; when $x=-4$, $y=2$, and $z=-7$.

61. $2a+5b-16.3$; when $a=5$ and $b=-6$.

Find (a) the sum of the numbers; (b) their difference; (c) their product; (d) the quotient of the first divided by the second.

62. -60 and -15. **63.** 0 and -14. **64.** -12 and 4.

65. 6 and -3. **66.** -52 and -13. **67.** 0 and 23.

18. Real number scale

On the horizontal line in Figure 1, we select a point O, called the **origin,** and we decide to let this point represent the number 0. We select a unit of length for measuring distances on the line. Then, if P is a positive number, we let it be represented by the point on

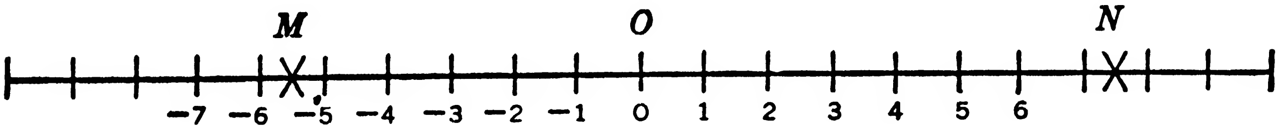

Fig. 1

the line which is at P units of distance from O to the right. The negative number $-P$ is represented by the point which is at P units of distance from O to the left. The points at whole units of distance from O represent the positive and negative integers. The other points on the line represent the numbers which are not integers. Thus, all real numbers are identified with points on the *real number scale* in Figure 1. If M is any real number, it can be thought of as a measure of the *directed distance* from O to M on the scale, where OM is considered *positive* when the direction from O to M is to the *right* and *negative* when the direction from O to M is to the *left.*

19. The less than and greater than relationships

All real numbers can be represented *in order from left to right* on the scale in Figure 1. We then say that one number * M is *less than* a second number N, or that N is *greater than* M, in case M is to the *left* of N on the number scale. We use the inequality signs $<$ and $>$ to represent *less than* and *greater than,* respectively. This definition includes as a special case the similar notion used in arithmetic for positive numbers. The present definition applies to all real numbers, positive, negative, or zero. If $a \neq b$,† then either $a < b$ or $a > b$.

ILLUSTRATION 1. In each of the following inequalities, we verify the result by placing the numbers on the scale in Figure 1. Thus, $-7 < -3$ because -7 is situated to the left of -3 in Figure 1. We read this inequality as "-7 *is less than* -3."

$$4 < 6; \qquad 0 < 8; \qquad -4 < 0; \qquad -6 < 5.$$

To say that $P > 0$ is equivalent to saying that P is *positive,* because the numbers to the *right* of O in Figure 1 are positive. To say that $M < 0$ is equivalent to saying that M is a negative number.

* Until otherwise indicated, the word *number* will refer to any real number.
† We read $\neq$ as "*not equal.*"

20. Numerical inequality

We say that one number b is *numerically less* than a second number c in case *the absolute value of b is less than the absolute value of c.* To distinguish this relation from *ordinary* inequality, we sometimes place the word *algebraically* before *greater than* or *less than* when they are used in the ordinary sense.

ILLUSTRATION 1. 5 is *numerically* less than 9 because $|5| < |9|$. It is also true that 5 is *algebraically* less than 9, because $5 < 9$.

ILLUSTRATION 2. We see that -3 is numerically less than -7 because $|-3| = 3$ and $|-7| = 7$, and $3 < 7$. On the other hand, $-7 < -3$. Thus, -3 is *algebraically greater* than -7 but *numerically less* than -7.

In Illustration 2, we observe a special case of the fact that, if one negative number b is *algebraically less* than a second negative number c, then b is *numerically greater* than c.

EXERCISE 4

Construct a real number scale 10 *inches long, and mark the locations of the positive and negative integers from* -10 *to* $+10$, *inclusive. Then, read each inequality and verify it by marking the two numbers on your scale.*

1. $5 < 9$. **2.** $0 < 7$. **3.** $-3 < 0$.

4. $-3 < 8$. **5.** $-5 < 2$. **6.** $-7 < -4$.

7. $8 > -9$. **8.** $5 > -3$. **9.** $-3 > -10$.

Mark the numbers in each problem on your scale and decide which sign, $<$ *or* $>$, *should be placed between the numbers.*

10. 7 and 9. **11.** -2 and 5. **12.** 0 and 8.

13. -6 and 0. **14.** -3 and 3. **15.** -2 and -7.

16. -9 and 10. **17.** 7 and 5. **18.** 8 and -3.

19. -6 and -3. **20.** -7 and -9. **21.** 8 and -10.

Read the inequalities and verify that they are correct.

22. $|-7| < |8|$ and $-7 < 8$. **23.** $|4| < |9|$ and $4 < 9$.

24. $|-3| > |0|$ but $-3 < 0$. **25.** $|-8| > 3$ but $-8 < 3$.

26. $|-5| < |-7|$ but $-5 > -7$.

27. $|-6| < |-9|$ but $-6 > -9$.

State which number (a) is algebraically greater than the other; (b) is numerically greater than the other.

28. -8; 6. **29.** -5; -3. **30.** 7; 4.

31. 0; -3. **32.** 5; 0. **33.** -2; 7.

34. -3; -7. **35.** 2; -6. **36.** -8; -3.

21. Signs of grouping

Parentheses, ( ), brackets, [ ], braces, { }, and the vinculum, $\overline{\quad}$, are symbols of grouping used to indicate terms whose sum is to be treated as a single number expression. Any general remark about *parentheses* which follows will be understood to apply as well to any other symbol of grouping.

Parentheses are useful for enclosing and separating algebraic expressions written side by side as an indication that they are to be multiplied.

ILLUSTRATION 1. $-3(-5)$ means -3 *times* -5, or $+15$.

In reading algebraic expressions, the student may use the words "*the quantity*" as he comes to the left marker for any symbol of grouping enclosing more than one term.

ILLUSTRATION 2. $(3-5a)(2+6a)$ is read "*the quantity* $3-5a$ *times the quantity* $2+6a$." If $a=4$,

$$(3-5a)(2+6a) = (3-20)(2+24) = (-17)(26) = -442.$$

Parentheses can be employed to avoid ambiguity in regard to the order of application of the fundamental operations.

ILLUSTRATION 3. Doubt arises as to the meaning of $(9-6\div 3)$. Does it mean $(9-6)\div 3$, which equals 1, or does it mean $9-(6\div 3)$, which is $(9-2)$ or 7? The two possible meanings were written without ambiguity.

ILLUSTRATION 4. If $a=2$, $b=-3$, and $c=5$, then

$$\begin{aligned} 5-3ab+2ac &= 5-[3(2)(-3)]+[2(2)(5)] \\ &= 5-(-18)+20 = 5+18+20 = 43. \end{aligned}$$

A factor multiplying a sum within parentheses should be used to multiply each term of the sum.

ILLUSTRATION 5. $$-3(2a) = (-3)(2)(a) = -6a.$$

ILLUSTRATION 6.

$$-4(a-2b+7) = -4a-4(-2b)-4(7) \quad (1)$$
$$= -4a+8b-28. \quad (2)$$

The student should write (2) without writing the right-hand side of (1).

ILLUSTRATION 7.

$$-(-5+3x-y) = (-1)(-5+3x-y)$$
$$= 5-3x+y.$$

The presence of a plus sign or a minus sign before any algebraic expression indicates that it is to be multiplied by $+1$ or -1, respectively. Multiplication by $+1$ would leave a sum *unchanged*, and multiplication by -1 would *change the signs* of all its terms. These remarks justify the following rules.

I. *To remove or to insert parentheses preceded by a plus sign, rewrite the included terms unchanged.*

II. *To remove or to insert parentheses preceded by a minus sign, rewrite the included terms with their signs changed.*

ILLUSTRATION 8.

$$+(3x-7+5y) = 3x-7+5y.$$
$$-(-2+5x-7y) = (-1)(-2+5x-7y) = 2-5x+7y.$$

ILLUSTRATION 9. In the sum on the left-hand side below, we enclose all terms after the first in parentheses preceded by a minus sign.

$$5a-2c+3d-8 = 5a-(2c-3d+8).$$

The signs of the terms enclosed were *changed* in order that, if the parentheses were *removed*, the original terms would be obtained.

In performing an operation which removes parentheses, if only explicit numbers are involved, it is best to compute each sum within parentheses before they are removed.

ILLUSTRATION 10.

$$-3(2-5+7) = -3(4) = -12.$$

EXERCISE 5

Compute the expression.

1. $(-2)(-5)$. **2.** $7(-3)$. **3.** $(-6)(4)$.

4. $(0)(-3)$. **5.** $-(-5)$. **6.** $+(-4)$.

7. $2(-5+9)$. **8.** $-3(5-14)$. **9.** $-4(-5-6)$.

10. $(-8+6)(-5+12)$. **11.** $(7-3)(16-5-2)$.

12. Compute $(3b - 4ac - 7)$ if $b = -2$, $a = 3$, and $c = -5$.

13. Compute $(-b + 3ac - 6c)$ if $a = 3$, $b = -4$, and $c = -2$.

14. Compute $(3 - 5a)(-2 + 6a)$ when $a = 4$.

Compute $(c - 2d)(4d - 3c)$ with the given values for c and d.

15. $c = 4$; $d = 3$. **16.** $c = -2$; $d = 5$. **17.** $c = -2$; $d = -3$.

Rewrite, by performing any indicated multiplication and removing parentheses. Evaluate, if no letters are present.

18. $-(17 - 8 - 3)$. **19.** $-(2 - 5 + 16)$. **20.** $+(3 - 6 + 15)$.

21. $-(2a - 5b + c)$. **22.** $+(-3 + 7a - b)$. **23.** $+(31 - 5a + y)$.

24. $-(-3 - 5x + 4y)$. **25.** $-(8a - 3b - c)$. **26.** $-(2x - 5y - 9)$.

27. $3(5a)$. **28.** $-2(4c)$. **29.** $-3(-5a)$. **30.** $5(-3x)$.

31. $4(2a - 3)$. **32.** $-2(x - y + 8)$. **33.** $-5(3 - a - 6c)$.

34. $2(-5 + 7a - 4bc)$. **35.** $3(6 - 4a + 5b)$. **36.** $-4(-5 + 6b)$.

Rewrite, enclosing the three terms at the right in parentheses preceded by a minus sign.

37. $-5 + 7a - 4b$. **38.** $-6a + 4b - c$. **39.** $6 - 3x - 4y$.

40. $2a - 3 + 5b - c$. **41.** $16 - 4a - b + 3c$.

42. $-13 + 5 - c + ab$. **43.** $2ac + 3 - 5a + 4c$.

44. Compute $(5 - 17) \div 3$; $5 - (17 \div 3)$.

22. Similar terms

Two terms such as $5abc$ and $7abc$, where the literal parts are the same, are called **similar terms** or **like terms**. In a term such as $5abc$, the explicit number which is a factor is called the **numerical coefficient** of the term, or, for short, **the coefficient.**

ILLUSTRATION 1. The numerical coefficient of $5abc$ is 5; of $-7abc$ is -7; of abc is 1. We never write the coefficient when it is 1. Thus, we would never write $1abc$ for abc.

A sum of similar terms can be collected (added) into a single term, by use of the distributive property of multiplication.

To collect a sum of similar terms, add their numerical coefficients and multiply the result by the common literal part.

ILLUSTRATION 2. $5ab + 7ab = ab(5 + 7) = 12ab.$

If we think of "ab" as a concrete object, the result here is obvious: 5 of the objects plus 7 of them equals 12 of them.

ILLUSTRATION 3. $-9ab + 4ab = ab(-9 + 4) = ab(-5) = -5ab.$

In finding a sum of algebraic expressions, we collect similar terms. A direction *to collect terms* means to collect *similar* terms.

EXAMPLE 1. Find the sum and the difference of

$$3a - 5y - 8 \quad \text{and} \quad 3y - 2a - 6.$$

FIRST SOLUTION. 1. The sum is $(3a - 5y - 8) + (3y - 2a - 6)$

$$= 3a - 2a - 5y + 3y - 8 - 6 = a - 2y - 14.$$

The student should learn to omit the intermediate details in such a solution.

2. The difference is

$$(3a - 5y - 8) - (3y - 2a - 6) = 3a - 5y - 8 - 3y + 2a + 6$$
$$= 3a + 2a - 5y - 3y - 8 + 6 = 5a - 8y - 2.$$

SECOND SOLUTION. To find the sum, arrange the given expressions with like terms in separate columns, and add. To find the difference, *change the signs* in $(3y - 2a - 6)$, or take its *negative*, and *add* similarly.

$$\textit{Add:}\ \begin{cases} 3a - 5y - 8 \\ \underline{-2a + 3y - 6} \end{cases} \qquad\qquad \textit{Add:}\ \begin{cases} 3a - 5y - 8 \\ \underline{2a - 3y + 6} \end{cases}$$
$$\textit{Sum} = a - 2y - 14. \qquad\qquad \textit{Difference} = 5a - 8y - 2.$$

In finding the difference, we could have changed the signs of $(-2a + 3y - 6)$ mentally without rewriting at the right. Thus, from the columns for the sum: *subtracting* $-2a$ *from* $3a$ *gives* $5a$; etc.

EXAMPLE 2. Perform indicated multiplications, remove parentheses, and collect terms:

$$3(4a - 3xy - 5b - 2) - 2(-5 + 3a + 5b - 6xy). \qquad (1)$$

SOLUTION. The sum equals

$$12a - 9xy - 15b - 6 + 10 - 6a - 10b + 12xy$$
$$= 6a + 3xy - 25b + 4. \qquad (2)$$

Comment. We can obtain a relatively certain check on the work by substituting values for the letters in (1) and (2). The results should be equal. In checking by this method, avoid substituting 1 for any letter.

CHECK. Substitute $a = 2$, $b = 4$, $x = 3$, and $y = 4$:

In (1): $3(8 - 36 - 20 - 2) - 2(-5 + 6 + 20 - 72) = -48.$

In (2): $12 + 36 - 100 + 4 = 52 - 100 = -48.$ *This checks.*

23. Nests of grouping signs

If a symbol of grouping encloses one or more other symbols of grouping, remove them by removing the *innermost symbol first*, and so on until the last one is removed. Usually, we enclose parentheses within brackets, and brackets within braces.

ILLUSTRATION 1. $-[3y-(2x-5+z)]$
$$= -[3y-2x+5-z] = -3y+2x-5+z.$$

ILLUSTRATION 2. $-[16-(2-7)-(-3+5)]$
$$= -[16-(-5)-(2)] = -[16+5-2] = -19.$$

EXERCISE 6

Collect similar terms.

1. $3a+8a.$
2. $-5b+7b.$
3. $-13x-5x.$
4. $-19ab+5ab.$
5. $-2cd+8cd.$
6. $5xy-5xy.$
7. $3x-a+2x-5a.$
8. $-3a+4b+7a-9b.$
9. $-2a-3c+5a-8c.$
10. $6x-4y+12x+3y.$
11. $11c-5cd+8c-13cd.$
12. $6h-5kw-11h+7kw.$

(a) Add the two expressions; (b) subtract the lower one from the upper one. After each operation, check by substituting convenient values for the letters in the given expressions and the final result.

13. $\begin{array}{r}-4a+7b-3\\ 2a-9b+7\\ \hline\end{array}$

14. $\begin{array}{r}3a-5b+5\\ -9a-4b-7\\ \hline\end{array}$

15. $\begin{array}{r}3x-5ab-\ c\\ -4x+6ab-3c\\ \hline\end{array}$

16. $\begin{array}{r}-2a-4bc-2d\\ 5a-6bc+\ d\\ \hline\end{array}$

17. $\begin{array}{r}3m-5k-\ h\\ -6m+4k-5h\\ \hline\end{array}$

18. $\begin{array}{r}-7r+3s-6\\ 8r-5s+9\\ \hline\end{array}$

Find the sum of the three expressions separated by semicolons. First arrange the expressions with like terms in separate columns.

19. $3x-2y-5;\ -5x+7y-8;\ -12x-8y+13.$
20. $-3+a-bc;\ 2bc-3a+5;\ 4a-5bc-9.$
21. $-2ac-5xy-8b;\ 4xy-3ac+5b;\ 7b-6xy-4ac.$
22. $7ad-5b-4a;\ 6a-3ad+5b;\ 3a+5ad-9b.$

Remove parentheses and collect similar terms.

23. $2(a - 3b) - 5(b - 2a) + 3(-a - 3b)$.

24. $5(x - 3y + 5) - 2(x + 2y - 4) - 3(-4x - y + 6)$.

25. $6(2a - h - k) - 3(a - 4h + 5k) + 2(4h - 3a - k)$.

By use of parentheses, write an expression for the quantity described; then remove parentheses and collect terms.

26. Subtract $3a - 2b$ and $2a + 5b$ from $-a - 5b$.

HINT. The result equals $(-a - 5b) - (3a - 2b) - (2a + 5b)$.

27. Subtract $2a - 3y - 5$ from $5a + 7y - 8$.

28. Subtract $3a - 7h$ and $5h - 6a$ from the sum of $2a - 3h$ and $5a + 4h$.

29. Multiply $3a - 5y - 3$ by 2, and $-5a + 7y - 5$ by -3, and then add the results.

Remove all signs of grouping and collect terms.

30. $-[4a - (2a + 3)]$.

31. $-[2t + (3 - 4t)]$.

32. $-(a - 2) - [2a - (a - 3)]$.

33. $a + [b + (a - b)]$.

34. $9 - [z - (6 - 2z)]$.

35. $2r + [r - (s + 4r)]$.

36. $x - [2x - (y + 3z)]$.

37. $2y - 5 + [3 - 2(y - 2)]$.

38. $3a - [3a - 4(5 - a)]$.

39. $3 - \{2x - 2[x - (2x - 5)]\}$.

40. $-\{a - [a - (2a - 7)]\}$.

41. $-\{2b - [6 - (3b - 4)]\}$.

42. $-3(2x - 3y) + 2[2 - (5x - y)] - (x + 7y - 8)$.

43. $2(3h - k - 5) - 3[h - 2(k - 3)] - 4(h - 2k + 2)$.

CHAPTER 2

INTRODUCTION TO FRACTIONS AND EXPONENTS

24. Fractions in lowest terms

The basic properties of fractions as met in arithmetic are primarily consequences of the definition of division. These properties extend immediately to fractions as met in algebra, where the only essential new feature is the introduction of negative numbers in the fractions. We shall recall and use the properties of fractions without rehearsing the sequence of definitions and proofs which, logically, would be necessary in building a foundation for the use of fractions in algebra.

FUNDAMENTAL PRINCIPLE. *The value of a fraction is not altered if both numerator and denominator are multiplied, or divided, by the same number, not zero.*

ILLUSTRATION 1. $\frac{5}{7} = \frac{5 \times 3}{7 \times 3} = \frac{15}{21}$. $\qquad \frac{a}{b} = \frac{ac}{bc}$.

$$\frac{36}{84} = \frac{36 \div 12}{84 \div 12} = \frac{3}{7}.$$

ILLUSTRATION 2. On multiplying numerator and denominator by -1 below, we obtain

$$\frac{-3}{-4} = \frac{(-1)(-3)}{(-1)(-4)} = \frac{3}{4}.$$

We say that a fraction is in *lowest terms* if its numerator and denominator have no common factor except $+1$ and -1.

To reduce a fraction to lowest terms, divide numerator and denominator by all their common factors.

ILLUSTRATION 3. $\frac{5acx}{7acy} = \frac{5x}{7y}$. (Divide out ac)

ILLUSTRATION 4. $$\frac{210}{135}=\frac{\cancel{5}\times\cancel{3}\times 7\times 2}{\cancel{5}\times\cancel{3}\times 3\times 3}=\frac{7\times 2}{3\times 3}=\frac{14}{9}.$$

In the preceding line we divided out the factor 5 and one 3 from numerator and denominator.

25. Change in sign for a fraction

If the numerator or denominator of a fraction is multiplied by -1, the sign before the fraction must be *changed*. These actions are equivalent to multiplying the fraction by *two* factors -1, whose product is $+1$. This keeps the value of the fraction unaltered.

ILLUSTRATION 1. $$\frac{a-3}{2}=-\frac{(-1)(a-3)}{2}=-\frac{3-a}{2}.$$

26. Multiplication and division of fractions

To multiply one fraction by another, multiply the numerators for a new numerator and multiply the denominators for a new denominator.

ILLUSTRATION 1. $$\frac{3}{5}\times\frac{6}{7}=\frac{18}{35}.\qquad \frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}.$$

To divide one fraction by another, invert the divisor and multiply the dividend by this inverted divisor.

ILLUSTRATION 2. $$\frac{4}{5}\div\frac{3}{7}=\frac{4}{5}\cdot\frac{7}{3}=\frac{28}{15}.$$

$$\frac{a}{b}\div\frac{c}{d}=\frac{\frac{a}{b}}{\frac{c}{d}}=\frac{a}{b}\cdot\frac{d}{c}=\frac{ad}{bc}.$$

It is frequently useful to recall that any number can be expressed as a fraction whose denominator is 1. By use of this fact we verify the following results.

To multiply a fraction by a number, multiply the numerator by the number.

To divide a fraction by a number, multiply the denominator by the number.

ILLUSTRATION 3. $$7\left(\frac{5}{6}\right)=\frac{7}{1}\cdot\frac{5}{6}=\frac{35}{6}.\qquad \frac{7}{5}\div 4=\frac{\frac{7}{5}}{\frac{4}{1}}=\frac{7}{5}\cdot\frac{1}{4}=\frac{7}{20}.$$

ILLUSTRATION 4. $$5 \div \frac{6}{7} = \frac{5}{1} \div \frac{6}{7} = \frac{5}{1} \cdot \frac{7}{6} = \frac{35}{6}.$$

ILLUSTRATION 5. $$c\left(\frac{a}{b}\right) = \frac{c}{1} \cdot \frac{a}{b} = \frac{ac}{b}.$$

$$\frac{a}{b} \div c = \frac{a}{b} \div \frac{c}{1} = \frac{a}{b} \cdot \frac{1}{c} = \frac{a}{bc}.$$

EXERCISE 7

Express the result as a fraction in lowest terms without a minus sign in numerator or denominator.

1. $\frac{18}{30}$. **2.** $\frac{32}{72}$. **3.** $\frac{45}{21}$. **4.** $\frac{66}{77}$. **5.** $\frac{5}{35}$.

6. $\frac{21}{14}$. **7.** $\frac{30}{120}$. **8.** $\frac{57}{38}$. **9.** $\frac{63}{81}$. **10.** $\frac{42}{63}$.

11. $\frac{-5}{3}$. **12.** $\frac{4}{-7}$. **13.** $\frac{-3}{-2}$. **14.** $\frac{-15}{35}$. **15.** $-\frac{78}{-26}$.

16. $\frac{4cd}{16c}$. **17.** $\frac{21x}{28xy}$. **18.** $\frac{3hy}{6y}$. **19.** $\frac{27a}{6ab}$.

20. $\frac{-bc}{3c}$. **21.** $\frac{4a}{-3a}$. **22.** $-\frac{-2}{6b}$. **23.** $\frac{5ad}{-3d}$.

24. $\frac{3}{5} \cdot \frac{4}{7}$. **25.** $\frac{4}{5} \cdot \frac{3}{8}$. **26.** $\frac{15}{7} \cdot \frac{21}{4}$. **27.** $\frac{b}{2} \cdot \frac{8}{d}$.

28. $\frac{3}{2} \div \frac{4}{5}$. **29.** $\frac{11}{7} \div \frac{22}{5}$. **30.** $\frac{15}{4} \div \frac{5}{6}$. **31.** $\frac{21}{38} \div \frac{14}{57}$.

32. $5\left(\frac{3}{8}\right)$. **33.** $21\left(\frac{2}{7}\right)$. **34.** $a\left(\frac{5}{9}\right)$. **35.** $c\left(\frac{3}{5}\right)$.

36. $\frac{5}{7}(4)$. **37.** $\frac{8}{15}(6)$. **38.** $\frac{8}{15} \div 2$. **39.** $\frac{3}{5} \div 6$.

40. $\frac{12}{7} \div 15$. **41.** $\frac{12}{7} \div b$. **42.** $\frac{14}{15} \div 2a$. **43.** $\frac{cd}{7} \div d$.

44. $\frac{3}{5} \cdot \frac{2}{7} \cdot \frac{5}{4}$. **45.** $\frac{3a}{4} \cdot \frac{5b}{6} \cdot \frac{2c}{9a}$. **46.** $\frac{x}{y} \cdot \frac{2c}{d} \cdot \frac{3d}{5x}$.

47. $\dfrac{\frac{3}{2}}{\frac{15}{4}}$. **48.** $\dfrac{\frac{4}{15}}{\frac{7}{3}}$. **49.** $\dfrac{\frac{8}{9c}}{\frac{4d}{5c}}$. **50.** $\dfrac{\frac{12a}{5b}}{\frac{8a}{15}}$.

51. $\dfrac{\frac{15}{7}}{6}$.   52. $\dfrac{\frac{14}{5}}{10}$.   53. $\dfrac{\frac{3h}{k}}{6}$.   54. $\dfrac{\frac{4w}{9d}}{2w}$.

55. $6 \div \frac{3}{2}$.   56. $5 \div \frac{4}{7}$.   57. $5 \div \frac{3}{10}$.   58. $5d \div \frac{3d}{c}$.

59. $\dfrac{3}{\frac{15}{7}}$.   60. $\dfrac{21}{\frac{14}{9}}$.   61. $\dfrac{2a}{\frac{c}{d}}$.   62. $\dfrac{5w}{\frac{15w}{8}}$.

63. $\frac{2}{5}\cdot\frac{2}{5}\cdot\frac{2}{5}$.   64. $\left(-\frac{3}{4}\right)\left(-\frac{3}{4}\right)\left(-\frac{3}{4}\right)\left(-\frac{3}{4}\right)$.   65. $\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)$.

27. Positive integral exponents

We write a^2 to abbreviate $a \cdot a$, and a^3 for $a \cdot a \cdot a$. We call a^2 the **square** of a and a^3 the **cube** of a.

ILLUSTRATION 1. $5^2 = 5 \cdot 5 = 25.$ $5^3 = 5 \cdot 5 \cdot 5 = 125.$

If m is any positive integer,* we define a^m by the equation

$$a^m = a \cdot a \cdot a \cdots a. \quad (m \text{ factors } a) \tag{1}$$

We call a^m the mth **power** of the base a and call m the **exponent** of this power. The exponent tells how many times the base is used as a factor.

ILLUSTRATION 2. $3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 81.$ $(-4)^2 = (-4)(-4) = +16.$

$$(-4)^3 = (-4)(-4)(-4) = -64. \qquad \left(\frac{2}{3}\right)^3 = \frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3} = \frac{8}{27}.$$

We notice that, by the laws of signs, an *even* power of a negative number is *positive* and an *odd* power is *negative.*

By definition, $b^1 = b$. Hence, when the exponent is 1 we usually omit it. Thus, 5 means 5^1 and y means y^1.

28. Index laws

The following laws for the use of exponents are called *index laws.* At present we will illustrate them, and verify their truth in special cases. The proofs of the laws will be given later.

* Until otherwise specified, any literal number used in an exponent will represent a positive integer.

I. *In multiplying two powers of the same base, add the exponents*

$$a^m a^n = a^{m+n}.$$

ILLUSTRATION 1. $a^3a^2 = a^{3+2} = a^5,$ because

$$a^3a^2 = (a \cdot a \cdot a)(a \cdot a) = a \cdot a \cdot a \cdot a \cdot a = a^5.$$

$$xx^2x^4 = x^1x^2x^4 = x^{1+2+4} = x^7.$$

II. *In obtaining a power of a power, multiply the exponents:*

$$(a^m)^n = a^{mn}.$$

ILLUSTRATION 2. $(a^3)^2 = a^{(3\times 2)} = a^6,$ because

$$(a^3)^2 = a^3 \cdot a^3 = (a \cdot a \cdot a)(a \cdot a \cdot a) = a \cdot a \cdot a \cdot a \cdot a \cdot a = a^6.$$

III. *To obtain a power of a product, raise each factor of the produc to the specified power and multiply:*

$$(abc)^n = a^nb^nc^n.$$

ILLUSTRATION 3. $(ab)^3 = a^3b^3,$ because

$$(ab)^3 = ab \cdot ab \cdot ab = (a \cdot a \cdot a)(b \cdot b \cdot b) = a^3b^3.$$

IV. *To obtain a power of a fraction, raise the numerator and denom inator to the specified power and divide:*

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

ILLUSTRATION 4. $\left(\frac{x}{y}\right)^4 = \frac{x^4}{y^4}$ because $\left(\frac{x}{y}\right)^4 = \frac{x}{y} \cdot \frac{x}{y} \cdot \frac{x}{y} \cdot \frac{x}{y} = \frac{x^4}{y^4}.$

ILLUSTRATION 5. $\left(\frac{2}{3}\right)^4 = \frac{2^4}{3^4} = \frac{16}{81}.$ (Law IV

$$\left(-\frac{3}{5}\right)^3 = \left[(-1)\left(\frac{3}{5}\right)\right]^3 = (-1)^3\left(\frac{3}{5}\right)^3 = -\frac{3^3}{5^3} = -\frac{27}{125}.$$

If we recall that an odd power of a negative number is negative, we ma abbreviate the preceding line by omitting the first two steps.

ILLUSTRATION 6. $(2a^2b)^3 = 2^3(a^2)^3b^3 = 8a^6b^3.$

ILLUSTRATION 7.

$$\left(\frac{3x^2y}{h^3}\right)^4 = \frac{(3x^2y)^4}{(h^3)^4} = \frac{3^4(x^2)^4y^4}{h^{12}} = \frac{81x^8y^4}{h^{12}}.$$ (Laws IV, II, III

EXERCISE 8

Compute by the definition of an exponent.

1. 2^4. **2.** 5^3. **3.** 10^2. **4.** 10^3. **5.** 10^4.

6. 10^5. **7.** $(-1)^2$. **8.** $(-1)^3$. **9.** $(-1)^4$. **10.** $(-5)^2$.

11. $(-3)^3$. **12.** $(-2)^5$. **13.** $(-5)^3$. **14.** 6^2. **15.** $(-3)^5$.

16. 10^6. **17.** $(-10)^3$. **18.** $(\frac{2}{3})^4$. **19.** $(\frac{3}{4})^3$. **20.** $(\frac{1}{5})^3$.

21. $(-\frac{1}{2})^3$. **22.** $(\frac{1}{10})^4$. **23.** -2^4. **24.** -3^3. **25.** -6^3.

26. $2(3^4)$. **27.** $3(-5^2)$. **28.** $6(10^3)$. **29.** $5(4^3)$. **30.** $-2(-5)^3$.

31. For what values of the positive integer n will $(-1)^n$ be positive and when will it be negative?

32. Compute $3a^2b$ if $a = -2$ and $b = 4$.

33. Compute $5xy^3$ if $x = 3$ and $y = -2$.

34. Compute $-4h^2k^3$ if $h = -3$ and $k = 2$.

35. If x is positive or negative, what is the nature of x^2, positive or negative? If x is negative, what is the nature of x^3 and of x^4?

Perform the indicated operation by use of the laws of exponents.

36. a^5a^4. **37.** z^2z^5. **38.** xx^3. **39.** y^3y^7.

40. $10^2 10^5$. **41.** xx^2x^4. **42.** yy^3y^7. **43.** b^3b^9b.

44. $3^3 3^5$. **45.** $a^h a^k$. **46.** $x^m x^{2n}$. **47.** $(a^2)^4$.

48. $(2^2)^3$. **49.** $(xy)^4$. **50.** $(cd)^3$. **51.** $(3x)^2$.

52. $(c^3)^5$. **53.** $(h^3)^2$. **54.** $(ax^2)^3$. **55.** $(3b)^4$.

56. $\left(\frac{c}{d}\right)^3$. **57.** $\left(\frac{h}{k}\right)^4$. **58.** $\left(\frac{3}{a}\right)^3$. **59.** $\left(\frac{x}{2}\right)^4$.

60. $(\frac{3}{2})^4$. **61.** $(\frac{3}{5})^3$. **62.** $(-\frac{2}{5})^4$. **63.** $(-\frac{1}{3})^5$.

64. $\left(\frac{ab}{x}\right)^3$. **65.** $\left(\frac{2h}{kw}\right)^4$. **66.** $\left(-\frac{3x}{yz}\right)^3$. **67.** $\left(-\frac{ac}{2b}\right)^4$.

68. $(x^2y^3)^4$. **69.** $(3c^2)^3$. **70.** $(2a^2b^4)^4$. **71.** $(3x^3w)^4$.

72. $(-2h^2)^3$. **73.** $(-3x^2)^4$. **74.** $(-5xy^2)^3$. **75.** $(-a^2x^3)^4$.

76. $\left(\frac{2d}{3b^2}\right)^4$. **77.** $\left(\frac{c^2x}{5a}\right)^3$. **78.** $\left(-\frac{3}{ax^2}\right)^3$. **79.** $\left(-\frac{x^2y^3}{2a}\right)^4$.

80. Find the value of $x^3 - 3x^2 + 4x - 7$ when (*a*) $x = 3$; (*b*) $x = -2$.

29. Integral rational terms

At present, in any sum, the typical term will be either an explicit number or the product of an explicit number and powers of literal numbers, where the exponents are positive integers. The explicit number is called the **numerical coefficient,** or for short *the coefficient* of the term. A term of this variety, or a sum of such terms, is said to be *integral and rational* * in the literal numbers.

An algebraic sum is called a **monomial** † if there is just *one* term, a **binomial** if there are just *two* terms, and a **trinomial** if there are just *three* terms. Any sum with *more than one* term also is called a **polynomial.**

ILLUSTRATION 1. $3x + 7ab$ is a binomial.

ILLUSTRATION 2. In the trinomial $-8 + x - 3ab^2$, the terms are -8, x, and $-3ab^2$, whose numerical coefficients are -8, 1, and -3, respectively.

ILLUSTRATION 3. $5x^2 - x - 7$ is an **integral rational polynomial** in x.

To multiply two integral rational terms, multiply their numerical coefficients and simplify the product of the literal parts by use of the law of exponents for multiplication.

ILLUSTRATION 4.
$$6ab^2(3a^2b^4) = (6)(3)(aa^2b^2b^4) = 18a^3b^6.$$
$$-3x^3y(4x^2y^6) = -12x^{3+2}y^{1+6} = -12x^5y^7.$$

To multiply a polynomial by a single term, multiply each term of the polynomial by the single term and form the sum of the results.

ILLUSTRATION 5.
$$5(3x^2 - 2x - 5) = 15x^2 - 10x - 25.$$
$$-2x^2y^3(3 - 7x^2y) = -6x^2y^3 - 2x^2y^3(-7x^2y) = -6x^2y^3 + 14x^4y^4.$$

EXERCISE 9

Perform the indicated multiplications and simplify the results by use of the law of exponents for multiplication.

1. $2x^2(5x^4)$. **2.** $3y(2y^5)$. **3.** $ab(3a)$. **4.** $xy^2(2x^2y)$.

5. $-5z(3z^7)$. **6.** $-2a^2(3a^9)$. **7.** $cd^2(4d^3)$. **8.** $x(-5x^3)$.

* The word *integral* refers to the fact that the exponents are *integers.* The force of the word *rational* will be pointed out later.

† This name need not be used very often because the simple word *term* is just as desirable.

9. $-4s(st)$. **10.** $3c(-c^3)$. **11.** $-2x(+xy^2)$. **12.** $-x^3(-2x^2)$.

13. $-ax^4(-2a^2x)$. **14.** $5xy(-2xy^2)$. **15.** $2a^3b(-4a^5b^2)$.

16. $-8m^3(-2m)$. **17.** $-4r^2h(-6rh^4)$. **18.** $-6c^2d^3(-3cd)$.

19. $3(4x-y)$. **20.** $-8(5x-2x^2)$. **21.** $-5(3-4a)$.

22. $4(2a-5b)$. **23.** $-3(-5x-4y)$. **24.** $x(2x^2-3x)$.

25. $2x(-3x-5x^3)$. **26.** $a(a^2-a^3)$. **27.** $2x^2(3-x^4)$.

28. $-4a(1-2ab)$. **29.** $-3h(k-hk)$. **30.** $ah^2(a-h^3)$.

31. $-5w(2-3w+4w^2)$. **32.** $-3hk^2(h^2-4hk-2k^2)$.

33. $3ab^2(2a^2b^3)(ab^4)$. **34.** $-2h^2k(-3hk)(-4h^3k^3)$.

35. $-4m^3n(-3m)(2mn^3)$. **36.** $4yz^2(-2y^2)(5y^3z)$.

37. $2x^ny(3x^hy^k)$. **38.** $2a^hb^n(-3a^2b^3)$. **39.** $-h^rk^s(3h^4k^3)$.

Multiply the polynomial by the term which is beneath it.

40. $\begin{array}{r} 5x^3-3x^2-2x-5 \\ 3x \\ \hline \end{array}$ **41.** $\begin{array}{r} 6-4x+5x^2-7x^3 \\ -2x^2 \\ \hline \end{array}$

42. $\begin{array}{r} a^2-2ab-b^2 \\ -2ab \\ \hline \end{array}$ **43.** $\begin{array}{r} 3-6ab-5a^2b^2-a^3b^3 \\ -3ab \\ \hline \end{array}$

44. $15(2x^2-\frac{1}{3}+\frac{2}{5})$. **45.** $8(\frac{3}{8}-\frac{1}{4}y+\frac{1}{2}y^2)$.

46. $12(\frac{2}{3}-\frac{3}{4}z+\frac{3}{2}z^2)$. **47.** $16(\frac{1}{4}a^2-\frac{5}{8}a-\frac{7}{8})$.

48. $\frac{2}{3}(12-6a+24a^2)$. **49.** $\frac{3}{4}(-16+8x-32x^2)$.

50. $\frac{2}{5}(10x^3-15x^2-\frac{5}{4}x-\frac{25}{6})$.

30. Multiplication of polynomials

To form the product of two polynomials, multiply one of them by each term of the other and collect similar terms.

ILLUSTRATION 1. $(2x-3y)(x^2-xy) = 2x(x^2-xy) - 3y(x^2-xy)$
$= 2x^3-2x^2y-3x^2y+3xy^2 = 2x^3-5x^2y+3xy^2.$

ILLUSTRATION 2. $(x+5)^2 = (x+5)(x+5)$
$= x(x+5)+5(x+5) = x^2+5x+5x+25 = x^2+10x+25.$

Before multiplying, if many terms are involved, arrange the polynomials in ascending (or descending) powers of one letter.

ILLUSTRATION 3. To multiply $(x^2 + 3x^3 - x - 2)(2x + 3)$:

$$
\begin{array}{lrl}
 & 3x^3 + \quad x^2 - \quad x - 2 & \\
(\textit{Multiply}) & 2x + 3 & \\
\hline
 & 6x^4 + \; 2x^3 - 2x^2 - 4x \qquad\quad & \text{(Multiplying by } 2x) \\
 & 9x^3 + 3x^2 - 3x - 6 & \text{(Multiplying by 3)} \\
\hline
(\textit{Add}) & 6x^4 + 11x^3 + \quad x^2 - 7x - 6 & = \textit{product.}
\end{array}
$$

EXAMPLE 1. Multiply $x^2 - y^2 + 2xy$ by $y^2 + x^2 - 2xy$.

SOLUTION.

$$
\begin{array}{l}
x^2 + 2xy - \; y^2 \ldots\ldots\ldots\ldots\ldots \\
x^2 - 2xy + \; y^2 \ldots\ldots\ldots\ldots\ldots \\
\hline
x^4 + 2x^3y - \; x^2y^2 \\
\quad - 2x^3y - 4x^2y^2 + 2xy^3 \\
\qquad\qquad\quad x^2y^2 + 2xy^3 - y^4 \\
\hline
x^4 \qquad\quad - 4x^2y^2 + 4xy^3 - y^4 \rightarrow
\end{array}
$$

CHECK. Place $x = 2$ and $y = -3$.

$= 4 - 12 - 9 = -17.$

$= 4 + 12 + 9 = 25.$

Hence, the product should equal

$25 \cdot (-17) = -425$

when $x = 2$ and $y = -3$.

$= 16 - 144 - 216 - 81 = -425.$

Comment. The preceding numerical check does not absolutely verify the result, but almost any error would cause the check to fail. In checking, if 1 were substituted for a letter, we could not detect an error in any of its exponents because every power of 1 is 1. Hence, in numerical checks, avoid substituting 1 for any letter.

EXERCISE 10

Multiply and collect similar terms. Check by substituting values for the letters, when directed by the instructor.

1. $(x - 3)(x + 4)$.
2. $(x + 9)(x + 10)$.

3. $(x - 5)(2x + 7)$.
4. $(x - 3)(7 - x)$.

5. $(5a - 7)(4a - 3)$.
6. $(3y - 2)(2 - 5y)$.

7. $(2h - 3k)(2h + 3k)$.
8. $(3w - 4r)(2w + 3r)$.

9. $(a + 3b)(2a - 5b)$.
10. $(x + y)(x - y)$.

11. $(3r - 5s)(3r + 5s)$.
12. $(a^2 - 2)(2a^2 - 5)$.

13. $(ab - 3)(2ab + 5)$.
14. $(3 - 5x^2)(2 + 3x^2)$.

15. $(cd - x)(cd + x)$.
16. $(ay - 2z)(2ay + 3z)$.

17. $(a + 3)^2$.
18. $(2x - y)^2$.
19. $(h - 4k)^2$.
20. $(4 + x)^2$.

21. $(3a - 2b)^2$.
22. $(2x + 4h)^2$.
23. $(ax - b)^2$.
24. $(5 - x^2y)^2$.

25. $(y^3 - 2)(y^3 + 5)$.
26. $(c^2 - 2a^2)(c^2 + 3a^2)$.

27. $(a^3 - b^2)(3a^3 + 4b^2)$.

28. $(x^2 - y^3)(2x^2 + y^3)$.

29. $(y - 2)(y^2 + 2y + 4)$.

30. $(4a^2 - 2a + 1)(2a + 1)$.

31. $(x - 3)(x^2 + 2x - 5)$.

32. $(a - 4)(3a^2 - 2a + 1)$.

33. $(2 + x)(3 - 4x - x^2)$.

34. $(c + 2)(2c - 5 - 3c^2)$.

35. $(1 - 2x)(2x - x^2 + 5)$.

36. $(h - 4)(2h^2 - 1 + h)$.

37. $(5 - b)(4 - 2b - b^2)$.

38. $(2 - y)(y + 5 - y^2)$.

39. $(3x^3 - 2x^2 + 5x - 7)(2x - 1)$.

40. $(2y^3 - 4y + 5y^2 - 3)(y + 3)$.

41. $(4y^3 - 3y + 5)(2y - 5)$.

42. $(y^3 + y + 2y^2 - 1)(y^2 - 2y + 3)$.

43. $(5 + 3x^2 - 4x)(5x^2 - 3 + x)$.

44. $(x^2 + 3x + 4)(x^2 + 3 - 2x)$.

45. $(5x - 2y + 3)^2$.

46. $(x + y)(2x^2 - 5xy - y^2)$.

47. $(a + b)(a^2 - ab + b^2)$.

48. $(x - 2y)(x^2 + 2xy + 4y^2)$.

49. $(x + 3)(x - 1)(2x - 5)$.

50. $(3x - y)(2x + y)(2x - y)$.

51. $(2a - 3)(3a + 5)(a - 2)$.

52. $(x - a^k)(x^2 + a^kx + a^{2k})$.

53. $(x^n - 3y^k)(x^{2n} + 3x^ny^k + 9y^{2k})$.

54. $(x^4 + y^4 - x^3y + x^2y^2)(x^2 - 2xy - y^2)$.

31. Exponents in division

By the definition of a power,

$$\frac{a^5}{a^3} = \frac{a \cdot a \cdot a \cdot a \cdot a}{a \cdot a \cdot a} = \frac{a \cdot a}{1} = a^2; \qquad \text{(Divide out } a \cdot a \cdot a\text{)}$$

$$\frac{a^3}{a^5} = \frac{a \cdot a \cdot a}{a \cdot a \cdot a \cdot a \cdot a} = \frac{1}{a \cdot a} = \frac{1}{a^2}. \qquad \text{(Divide out } a \cdot a \cdot a\text{)}$$

The preceding results illustrate the following index law.

In dividing one power of a specified base by another power of the base, subtract the exponents:

$$\frac{a^m}{a^n} = a^{m-n} \quad (\text{if } m > n); \qquad \frac{a^m}{a^n} = \frac{1}{a^{n-m}} \quad (\text{if } m < n). \qquad (1)$$

In addition to formulas 1, we note that

$$\frac{a^n}{a^n} = 1. \tag{2}$$

ILLUSTRATION 1. $\dfrac{a^5}{a^5} = 1.$ $\dfrac{x^4}{x^{10}} = \dfrac{1}{x^{10-4}} = \dfrac{1}{x^6}.$ $\dfrac{h^7}{h^4} = h^{7-4} = h^3.$

In dividing one integral rational term by another, we use formulas 1 and 2 to reduce the fraction to lowest terms.

ILLUSTRATION 2. In the following simplification, we can think of dividing numerator and denominator by $5ax^5$. Or, we can think of dividing numerator and denominator by 5 and, also, of applying formulas 1 to the powers of a and of x, separately:

$$\frac{-15a^3x^5}{10ax^9} = -\frac{3}{2}\cdot\frac{a^3}{a}\cdot\frac{x^5}{x^9} = -\frac{3a^{3-1}}{2x^{9-5}} = -\frac{3a^2}{2x^4}.$$

ILLUSTRATION 3. $$\frac{16a^3x^2}{-2ax^4} = \frac{16}{-2}\cdot\frac{a^3}{a}\cdot\frac{x^2}{x^4} = -\frac{8a^2}{x^2}.$$

ILLUSTRATION 4. $$-\frac{aw^5}{-2a^4w^2} = -\left(-\frac{1}{2}\right)\cdot\frac{a}{a^4}\cdot\frac{w^5}{w^2} = \frac{w^3}{2a^3},$$

where the intermediate details should be performed mentally.

ILLUSTRATION 5. $$\frac{-6a^3y^2}{-15a^3y^6} = \frac{-6}{-15}\cdot\frac{a^3}{a^3}\cdot\frac{y^2}{y^6} = \frac{2}{5y^4}.$$

32. Division by a single term

To divide a polynomial by a single term, divide each term of the polynomial by the single term and combine the results.

ILLUSTRATION 1. $$\frac{-3x^4 + 6x^2 - 9x}{-3x} = -\frac{3x^4}{-3x} + \frac{6x^2}{-3x} - \frac{9x}{-3x}$$
$$= x^3 - 2x + 3.$$

EXAMPLE 1. Perform the division: $(4a^2b^4 - 8a^2b - 2b^2) \div (-2ab^3)$.

SOLUTION. Write the quotient as a fraction:

$$\frac{4a^2b^4 - 8a^2b - 2b^2}{-2ab^3} = \frac{4a^2b^4}{-2ab^3} - \frac{8a^2b}{-2ab^3} - \frac{2b^2}{-2ab^3}$$
$$= -2ab + \frac{4a}{b^2} + \frac{1}{ab}.$$

EXERCISE 11

Perform the division, expressing the result as a fraction or sum of fractions in lowest terms by use of the law of exponents for division. Express each final fraction without a minus sign in numerator or denominator.

1. $\dfrac{y^3}{y}$. **2.** $\dfrac{a^8}{a^2}$. **3.** $\dfrac{x^3}{x^8}$. **4.** $\dfrac{h^9}{h^9}$. **5.** $\dfrac{x^3}{x^2}$.

6. $\dfrac{a}{a^6}$. **7.** $\dfrac{3^9}{3^7}$. **8.** $\dfrac{4^5}{4^8}$. **9.** $\dfrac{x^4}{x^5}$. **10.** $\dfrac{2^8}{2^3}$.

11. $\dfrac{8y^3}{4y}$. **12.** $\dfrac{-15ab}{5a}$. **13.** $\dfrac{12x^2}{3x}$. **14.** $\dfrac{15x^3}{-3x^2}$.

15. $-\dfrac{-21a^2}{3a}$. **16.** $\dfrac{3x^3}{12x^3}$. **17.** $\dfrac{5r}{25r^2}$. **18.** $-\dfrac{-2a^5}{-4a^2}$.

19. $-\dfrac{-2z^5}{6z}$. **20.** $-\dfrac{9b}{-27bc^3}$. **21.** $\dfrac{35a^4}{5a^7}$. **22.** $\dfrac{15w^3}{30w^3}$.

23. $\dfrac{x^4y^2}{xy^5}$. **24.** $\dfrac{a^5b^3}{a^6b}$. **25.** $\dfrac{h^2k^5}{h^3k^2}$. **26.** $\dfrac{x^2y^4z}{x^3yz^6}$.

27. $\dfrac{45s^2t^3}{5st^2}$. **28.** $\dfrac{-21z^3h^2}{3z^4h}$. **29.** $\dfrac{18b^2c}{-3bc^3}$. **30.** $\dfrac{36m}{18mn}$.

31. $\dfrac{-5d}{-20cd}$. **32.** $\dfrac{-24ht^2}{3ht^3}$. **33.** $\dfrac{-42x^4y}{-6x^3y}$. **34.** $\dfrac{-12ab}{-24a^2b^2}$.

35. $(8x^3y) \div (-24x^3y)$. **36.** $(-49r^2s^3) \div (-7rs^4)$.

37. $(-8a) \div (24ab^2)$. **38.** $(4ac) \div (72a^3c^2)$.

39. $(48x^2y^3) \div (-8xy^2)$. **40.** $(-36ab^2) \div (-4ab)$.

41. $\dfrac{6a + 20b}{4}$. **42.** $\dfrac{5h - 25k}{-5}$. **43.** $\dfrac{4a + 16b}{-4}$.

44. $\dfrac{7x^2 - 5x}{x}$. **45.** $\dfrac{3a^3 - 2a^4}{a^2}$. **46.** $\dfrac{5w^3 - 3w^5}{w^2}$.

47. $\dfrac{6a^2 - 15a^4}{-3a^2}$. **48.** $\dfrac{4y^3 - 8y^5}{2y^2}$. **49.** $\dfrac{8a^2h^4 - 16a^2h^3}{2a^2h^5}$.

50. $(8a^3 - 6a^2 - 4a) \div (-2)$. **51.** $(6x - 3x^2 - 9) \div (-3)$.

52. $(x^4 - 3x^3 - 5x^2) \div x^2$. **53.** $(y^2 - 5y - y^3) \div (-y)$.

54. $(-36 + 12b - 9b^2) \div 3$. **55.** $(x^4 - 3x^2 + 5x) \div (15x^4)$.

56. $(32a^2b^4 + 48a^3y^4) \div (16a^2)$. **57.** $(21a^2b^2 - 42ab^3) \div (-7ab^2)$.

58. $\dfrac{-ax^2 - 3ax + 5a}{ax}$.

59. $\dfrac{7x^3 - 4x^2 - 3x + 2}{-2x}$.

60. $\dfrac{5a^2b^2c - 7ab^2c^3 + 6c^4}{2abc^2}$.

61. $\dfrac{6x^2y^2 - 12xy^4 + y^6}{3xy^3}$.

62. $\dfrac{a^3b - 18a^2b^2 - 27b^3}{-3ab^2}$.

63. $\dfrac{abd^3 - a^2bd - a^3}{abd}$.

33. Fundamental equation of division

In the following long division, we use the customary terminology of arithmetic:

$$\left.\begin{array}{r|l} & \text{15 } (\textit{Quotient}) \\ \hline (\textit{Divisor})\ 17 & 259\ (\textit{Dividend}) \\ & \underline{17} \\ & 89 \\ & \underline{85} \\ & 4\ (\textit{Remainder}) \end{array}\right\} \tag{1}$$

$$259 = (17 \times 15) + 4; \qquad \frac{259}{17} = 15\tfrac{4}{17}. \tag{2}$$

In Section 11, when we met the notation $a \div b$, we called its *complete* value the *quotient* of a divided by b. In (1), the complete value of the quotient is $15\frac{4}{17}$. Hence, if there were danger of lack of clearness, in (2) we would call 15 the *partial* quotient. Frequently, the word *quotient* refers to a *partial* quotient. When appropriate, we shall use the qualifying word *partial* to prevent ambiguity.

After any step in a division process similar to that met in (1), the remainder and partial quotient satisfy the equation

$$\frac{\textbf{dividend}}{\textbf{divisor}} = \textbf{quotient} + \frac{\textbf{remainder}}{\textbf{divisor}}. \tag{3}$$

Or, $$\textbf{dividend} = (\textbf{quotient})(\textbf{divisor}) + \textbf{remainder}. \tag{4}$$

Equation 4 is frequently called the *fundamental equation of division.* The first equation in (2) is an illustration of (4).

34. Division by a polynomial

The long division process for polynomials is similar to long division in arithmetic. We say that the division is *exact* if the final remainder is *zero.*

EXAMPLE 1. Divide: $(x^2 + 3x - 40) \div (x - 5)$.

SOLUTION. $(x^2 \div x) = x$; this is the first term of the quotient. Then, $x(x - 5) = x^2 - 5x$; we subtract this from the dividend, obtaining $8x - 40$.

2. $(8x \div x) = 8$; this is the second term of the quotient. Then, $8(x - 5) = 8x - 40$. We subtract this, obtaining zero.

$$\begin{array}{rl}
 & x + 8 \ (\textit{Quotient}) \\
(\textit{Divisor})\ x - 5 & \overline{\big)\, x^2 + 3x - 40}\ (\textit{Dividend}) \\
(\textit{Subtract}) & \underline{x^2 - 5x} \\
 & \qquad 8x - 40 \\
(\textit{Subtract}) & \qquad \underline{8x - 40} \\
 & \qquad\qquad 0\ (\textit{Remainder})
\end{array}$$

CHECK. Since the division is exact, we obtain a check by verifying that

$$(\textit{quotient})(\textit{divisor}) = \textit{dividend}.$$

We find $(x + 8)(x - 5) = x^2 + 3x - 40$, which checks.

EXAMPLE 2. Divide: $(4x^3 - 9x - 8x^2 + 7) \div (2x - 3)$.

SOLUTION. 1. We first arrange the dividend in descending powers of x, obtaining

$$4x^3 - 8x^2 - 9x + 7.$$

2. Since $(4x^3 \div 2x) = 2x^2$, this is the first term of the quotient; etc., as follows.

$$\begin{array}{ll}
 & \quad 2x^2 - x - 6\ (\textit{Quotient}) \\
(\textit{Divisor})\ 2x - 3 & \overline{\big)\, 4x^3 - 8x^2 - 9x + 7}\ (\textit{Dividend}) \\
2x^2(2x - 3) \rightarrow (\textit{Subtract}) & \quad \underline{4x^3 - 6x^2} \\
[(-2x^2) \div 2x] = -x. & \qquad\ -2x^2 - 9x \\
-x(2x - 3) \rightarrow (\textit{Subtract}) & \qquad\ \underline{-2x^2 + 3x} \\
[(-12x) \div 2x] = -6. & \qquad\qquad -12x + 7 \\
-6(2x - 3) \rightarrow (\textit{Subtract}) & \qquad\qquad \underline{-12x + 18} \\
 & \qquad\qquad\qquad -11\ (\textit{Remainder})
\end{array}$$

Conclusion. From equation 3, Section 33.

$$\frac{4x^3 - 8x^2 - 9x + 7}{2x - 3} = 2x^2 - x - 6 - \frac{11}{2x - 3}.$$

CHECK. Substitute $x = 3$ in the dividend, divisor, and quotient:

dividend $= 16$; *divisor* $= 3$; *quotient* $= 9$; *remainder* $= -11$.

Refer to equation 4, Section 33: does

$$16 = 3(9) - 11 = 27 - 11 = 16?$$

Since this equality is satisfied, we conclude that the solution is correct.

SUMMARY. *To divide one polynomial by another:*

1. *Arrange each of them in either ascending or descending powers of some common letter.*

2. *Divide the first term of the dividend by the first term of the divisor and write the result as the first term of the quotient.*

3. *Multiply the whole divisor by the first term of the quotient and subtract the product from the dividend.*

4. *Consider the remainder obtained in Step 3 as a new dividend and repeat Steps 2 and 3; etc.*

Note 1. The numerical check in Example 2 does not constitute an absolute verification of the solution. To verify the result, we should (*without* substituting a special value for x) *multiply the divisor by the quotient, add the remainder, and notice if we thus obtain the dividend.*

EXERCISE 12

Divide and summarize as in the solution of Example 2, Section 34. If the division is exact, check by multiplying the divisor by the quotient. If the division is not exact, check by substituting convenient values for the literal numbers.

1. $(x^2 + 7x + 12) \div (x + 3)$.

2. $(y^2 + 15y + 36) \div (y + 12)$.

3. $(c^2 - 10c + 21) \div (c - 7)$.

4. $(d^2 - 12d + 35) \div (d - 5)$.

5. $(s^2 + 6s - 27) \div (s + 9)$.

6. $(2x^2 - 13x + 15) \div (x - 5)$.

7. $(y^2 + 6y - 40) \div (y + 10)$.

8. $(54 + 3x - x^2) \div (6 + x)$.

9. $(4c^2 - 15) \div (2c + 3)$.

10. $(6a^2 - ab - b^2) \div (3a + b)$.

11. $(x^4 + 3x^2 - 4) \div (x^2 - 1)$.

12. $(h^4 + 3h^2 - 10) \div (h^2 + 5)$.

13. $(2x^2 + 7x + 8) \div (x + 2)$.

14. $(3a^2 - 7) \div (a - 5)$.

15. $(2a^2 - ab - 6b^2) \div (2a + b)$.

16. $(6x^2 + xy - 2y^2) \div (3x + 2y)$.

17. $(4x^6 + 5x^3 - 6) \div (4x^3 - 3)$.

18. $(2a^6 - a^3 - 15) \div (2a^3 + 5)$.

19. $(5x + 3x^2 - 3) \div (x - 2)$.

20. $(a + 6a^2 + 3) \div (2a - 1)$.

21. $(x^3 + 3x^2 - 2x - 6) \div (x + 3)$.

22. $(5a - 2a^2 + 3a^3 - 26) \div (a - 2)$.

23. $(4x^3 - 9x - 8x^2 + 7) \div (2x - 3)$.

24. $(36y - 9 - 19y^2 - 15y^3) \div (3y^2 + 5y - 4)$.

25. $(8y^3 - 18y^2 - 6 + 11y) \div (4y^2 - 3y + 2)$.

26. $(2y^3 - 5y^2 - 12 + 11y) \div (2y - 3)$.

27. $(x^4 - 4x^3 + 3x^2 - 4x + 15) \div (x - 3)$.

28. $(8x^2 - 3 - 5x^3) \div (7x - 2 + 5x^2)$.

29. $(2x^3 + 9x^2y + 12y^3 + 17xy^2) \div (2x + 3y)$.

30. $(a^3 - 3a^2b - b^3 + 3ab^2) \div (a^2 - 2ab + b^2)$.

31. $(x^3 + 27) \div (x + 3)$.

32. $(a^3 + b^3) \div (a + b)$.

33. $(x^3 - y^3) \div (x - y)$.

34. $(16x^4 - y^4) \div (2x - y)$.

35. $\dfrac{8w^3 - 27}{2w - 3}$.

36. $\dfrac{x^3 - 8y^3}{x - 2y}$.

37. $\dfrac{a^6 - b^6}{a^2 - b^2}$.

38. $\dfrac{6x^4 + 7x^3 - 43x^2 - 9x - 5}{5x + 3x^2 - 7}$.

39. $\dfrac{6x^4 - 6z^2 - 5x^2z}{3x^2 + 2z}$.

40. $(6a^{3h} + 13a^{2h} - 4a^h - 15) \div (2a^h + 3)$.

35. Fractions with a common denominator

In a fraction, the bar should be thought of as a *vinculum*, a symbol of grouping, which encloses the numerator. We use this fact in adding fractions.

ILLUSTRATION 1. In the fraction $-\dfrac{3 - a}{5}$, the bar encloses $3 - a$ and the fraction equals $-(3 - a) \div 5$.

SUMMARY. *To express a sum of fractions with a common denominator as a single fraction:*

1. *Form the sum of the numerators, where each is enclosed in parentheses and is given the sign of its fraction.*

2. *Divide by the common denominator.*

ILLUSTRATION 2.
$$\frac{8}{5} - \frac{3}{5} + \frac{9}{5} = \frac{8 - 3 + 9}{5} = \frac{14}{5}.$$

ILLUSTRATION 3.
$$\frac{6}{11a} - \frac{5 - x}{11a} + \frac{3 - 2x}{11a}$$
$$= \frac{6 - (5 - x) + (3 - 2x)}{11a} = \frac{6 - 5 + x + 3 - 2x}{11a} = \frac{4 - x}{11a}.$$

36. Alteration of a denominator

To change a fraction to an equal one having an added factor in the denominator, we must multiply *both numerator and denominator* by this factor, in order to leave the value of the fraction unaltered.

ILLUSTRATION 1. To change $\frac{3}{7}$ to 14ths, we multiply numerator and denominator by 2 because $\frac{14}{7} = 2$:

$$\frac{3}{7} = \frac{3 \times 2}{7 \times 2} = \frac{6}{14}.$$

ILLUSTRATION 2. To change the following fraction to one where the denominator is $6a^3b$, we multiply numerator and denominator by $2a^2b$, because $6a^3b \div 3a = 2a^2b$:

$$\frac{5 - x}{3a} = \frac{2a^2b(5 - x)}{2a^2b(3a)} = \frac{10a^2b - 2a^2bx}{6a^3b}.$$

37. Prime integers

An integer is said to be *prime* if it has no factors except itself, and $+1$ and -1, which are factors of *any* algebraic expression. Two factors are considered essentially the same if they differ only in sign, and then their product can be expressed as a power of either one of them. *To factor* an integer will mean to express it as a product of powers of distinct *prime* factors.

ILLUSTRATION 1. 2, 3, 5, 7, 11, etc., are prime numbers.

ILLUSTRATION 2. When factored, $200 = 2 \cdot 2 \cdot 2 \cdot 5 \cdot 5 = 2^3 5^2$.

38. Lowest common multiple

At present, when we refer to a *monomial*, or *single term*, we shall mean an integral rational term whose numerical coefficient is an *integer*.

The *lowest common multiple*, **LCM**, of two or more monomials is defined as the term with *smallest* positive coefficient, and *smallest* exponents for the literal numbers, which has the given term as a factor. As a special case, the LCM of two or more integers is the *smallest positive integer* having each given integer as a factor.

ILLUSTRATION 1. The LCM of 3, 5, and 10 is 30.

ILLUSTRATION 2. The LCM of $3ab^3$ and $5a^2b$ is $15a^2b^3$.

SUMMARY. *To find the* LCM *of two or more terms:*

1. *Express each term with its coefficient factored.*

2. *Form the product of all letters in the terms and all the different prime factors of the coefficients, giving to each letter or integral factor the highest exponent it possesses in the given terms.*

EXAMPLE 1. Find the LCM of $20a^2b^3$ and $70a^4b$.

SOLUTION 1. In factored form, $20a^2b^3 = 2^2\cdot 5a^2b^3$; $70a^4b = 2\cdot 5\cdot 7a^4b$.

2. Hence, LCM $= 2^2\cdot 5\cdot 7a^4b^3 = 140a^4b^3$.

Note 1. In brief, the LCM of two or more terms equals *the product of the* LCM *of their coefficients and the highest powers of the letters seen in the terms.*

EXERCISE 13

Express the sum of fractions as a single fraction in lowest terms.

1. $\frac{3}{4}+\frac{7}{4}-\frac{9}{4}$.
2. $\frac{2}{5}-\frac{18}{5}+\frac{6}{5}$.
3. $\frac{2}{3}+\frac{b}{3}-\frac{a}{3}$.
4. $\frac{c}{8}-\frac{d}{8}-\frac{7}{8}$.
5. $\frac{3}{a}-\frac{5}{a}+\frac{7-b}{a}$.
6. $\frac{11}{z}-\frac{5}{z}+\frac{x+3}{z}$.
7. $\frac{3}{7}-\frac{2a-5b}{7}$.
8. $\frac{5}{3}-\frac{6-3a}{3}$.
9. $-\frac{x-2y}{8}-\frac{7}{8}$.
10. $-\frac{2b-3c}{5}+\frac{a}{5}$.
11. $\frac{3a-2}{b}-\frac{4a-5}{b}-\frac{2}{b}$.
12. $\frac{4y-3}{9}-\frac{5}{9}-\frac{2y-5}{9}$.
13. $\frac{3}{a^2b}-\frac{6x}{a^2b}-\frac{3-x}{a^2b}$.
14. $\frac{2h}{3c^2d}-\frac{3h-2}{3c^2d}$.
15. $\frac{5cd}{3x^2y}-\frac{2cd-3}{3x^2y}$.
16. $\frac{7h}{2ab^3}-\frac{4h-3}{2ab^3}$.

Write the missing numerator or denominator to create equality.

17. $\frac{3}{5}=\frac{\ }{10}$.
18. $\frac{5}{11}=\frac{\ }{77}$.
19. $\frac{3}{17}=\frac{6}{\ }$.
20. $\frac{5}{6}=\frac{\ }{6a}$.
21. $\frac{3}{7}=\frac{\ }{7bc}$.
22. $\frac{4}{9}=\frac{8x}{\ }$.

23. $\dfrac{2}{a^2b} = \dfrac{\quad}{a^4b^3}$. **24.** $\dfrac{5x}{2a^3b^2} = \dfrac{\quad}{2a^5b^3}$. **25.** $\dfrac{7h}{9x^4} = \dfrac{\quad}{36x^7}$.

Express the fraction as an equal one having the specified denominator.

26. $\dfrac{2}{7}$; denom., 21. **27.** $\dfrac{5}{8}$; denom., 32.

28. $\dfrac{4}{5}$; denom., 40. **29.** $\dfrac{c}{7}$; denom., 35.

30. $\dfrac{a}{b}$; denom., bc. **31.** $\dfrac{x}{2y}$; denom., $6y$.

32. $\dfrac{3}{5xy^2}$; denom., $10xy^3$. **33.** $\dfrac{a}{3x^2y^4}$; denom., $18x^3y^7$.

34. $\dfrac{c}{2hk^3}$; denom., $20h^2k^4$. **35.** $\dfrac{3d}{5a^4b}$; denom., $20a^5b^3$.

Find the LCM *of the given terms. As a partial check, verify that each term is a factor of the* LCM.

36. 5; 4; 10. **37.** 16; 24; 48. **38.** 12; 54; 30.

39. 15; 12; 75. **40.** 200; 36; 28. **41.** 300; 27; 21.

42. $6x^3y$; $9x^2y^2$; $15xy^2$. **43.** $8a^2b^5$; $4a^3b^4$; $6ab^6$.

44. $2ab$; $14a^2b^3$; $6b^4$. **45.** $6a^2x$; $4a^3x^2$; $9ax^4$.

46. $3x^2y^2$; $12xy^3$; $20x^3$. **47.** $5h^3k^2$; $10hk^3$; $16h^2k$.

48. Change $\frac{2}{3}$, $\frac{5}{7}$, and $\frac{3}{14}$ to the denominator 42, and then add the fractions.

39. Addition of fractions *

To combine a sum of fractions into a single fraction, the given fractions must be changed to new forms having a common denominator. We define the **lowest common denominator, LCD,** of a set of fractions as the LCM of their denominators.

ILLUSTRATION 1. The LCD of $\frac{1}{6}$, $\frac{3}{5}$, and $\frac{7}{8}$ is the LCM of 6, 5, and 8 or $3\cdot5\cdot8$ or 120. Hence, to add the fractions, we change them to new fractions whose denominator is 120:

$$\frac{1}{6}+\frac{3}{5}+\frac{7}{8}=\frac{20}{6\cdot20}+\frac{3\cdot24}{5\cdot24}+\frac{7\cdot15}{8\cdot15}=\frac{20+72+105}{120}=\frac{197}{120}.$$

* To simplify the present chapter, we will deal only with the case where each denominator is an integral rational term.

SUMMARY. *To express a sum of fractions as a single fraction:*

1. *Find the* LCD *of the given fractions.*

2. *For each fraction, divide the* LCD *by the denominator and then multiply numerator and denominator by the resulting quotient, to change to an equal fraction having the* LCD.

3. *Form the sum of the new numerators, where each one is enclosed in parentheses and is given the sign of its fraction.*

4. *Place the result of Step* 3 *over the* LCD, *remove parentheses in the numerator, and reduce the fraction to lowest terms.*

ILLUSTRATION 2. In the following sum the LCD is 20; we think of 2 as 2/1. We observe that $\frac{20}{4} = 5$ and $\frac{20}{10} = 2$. Hence,

$$2 - \frac{3x}{4} - \frac{7 - 2x}{10} = \frac{2\cdot 20}{20} - \frac{5(3x)}{20} - \frac{2(7 - 2x)}{20} \tag{1}$$

$$= \frac{40 - 15x - 2(7 - 2x)}{20} = \frac{40 - 15x - 14 + 4x}{20} = \frac{26 - 11x}{20}.$$

EXAMPLE 1. Express as a single fraction: $\dfrac{3y}{5a^3x} - \dfrac{7}{3ax^2}$.

SOLUTION. 1. The LCD is $15a^3x^2$. We have

$$\frac{15a^3x^2}{5a^3x} = 3x; \qquad \frac{15a^3x^2}{3ax^2} = 5a^2.$$

2. Hence, we multiply by $3x$ and $5a^2$ in the given fractions to change to the LCD:

$$\frac{3y}{5a^3x} - \frac{7}{3ax^2} = \frac{3y(3x)}{5a^3x(3x)} - \frac{7(5a^2)}{3ax^2(5a^2)} = \frac{9xy - 35a^2}{15a^3x^2}.$$

EXERCISE 14

Combine into a single fraction in lowest terms.

1. $\frac{5}{8} + \frac{3}{4}$. **2.** $\frac{3}{7} + \frac{9}{14}$. **3.** $\frac{5}{6} - \frac{1}{3}$. **4.** $\frac{3}{5} - \frac{9}{10}$.

5. $\frac{13}{16} - \frac{3}{8}$. **6.** $\frac{1}{5} + \frac{2}{3}$. **7.** $\frac{2}{7} + \frac{5}{21}$. **8.** $\frac{1}{6} - \frac{3}{5}$.

9. $\frac{a}{3} + \frac{b}{6}$. **10.** $\frac{2c}{5} - \frac{d}{2}$. **11.** $\frac{h}{3} - \frac{k}{4}$. **12.** $\frac{3r}{2} + \frac{s}{5}$.

13. $\frac{1}{2} - \frac{3}{5} + \frac{7}{8}$. **14.** $1 - \frac{3}{8} - \frac{4}{3}$. **15.** $\frac{3}{7} - 2 + \frac{5}{21}$.

16. $\frac{12}{7} - \frac{4}{5} + 3.$

17. $\frac{3}{10} - \frac{7}{30} + \frac{4}{5}.$

18. $-2 + \frac{1}{6} + \frac{4}{15}.$

19. $\frac{5h-3}{7} + \frac{3}{14}.$

20. $\frac{3a-b}{5} + \frac{7}{10}.$

21. $\frac{2}{9} - \frac{x-2y}{3}.$

22. $3 + \frac{2a-5}{4}.$

23. $2 - \frac{5x-y}{3}.$

24. $4 - \frac{3a-b}{5}.$

25. $\frac{a-3}{2} + \frac{b+2}{5}.$

26. $\frac{2x-3}{5} + \frac{3-5x}{35}.$

27. $\frac{3a}{5} - 2 - \frac{a+7}{25}.$

28. $4 - \frac{2x-3}{10} - \frac{x-7}{18}.$

29. $\frac{3}{4a} - \frac{5}{6a}.$

30. $\frac{b}{3x} - \frac{c}{2x}.$

31. $\frac{h}{4k} - \frac{w}{3k}.$

32. $\frac{3}{2a} - \frac{4}{3b}.$

33. $\frac{5}{12y} - \frac{3}{2y}.$

34. $\frac{7}{6x} - \frac{3}{8xy}.$

35. $\frac{5}{hk} - \frac{3}{kr}.$

36. $\frac{c}{x^2} - \frac{b}{2x}.$

37. $\frac{5}{a^2b} - \frac{3}{2ab^3}.$

38. $\frac{2}{3x^2} - \frac{5}{7xy^3}.$

39. $\frac{4}{3a^2} - \frac{5y-1}{5ab}.$

40. $\frac{3}{2xy} - \frac{4x-y}{4x^2y^3}.$

41. $\frac{-3a}{4} - 5 - \frac{2a-3}{10}.$

42. $-2 + \frac{6-2h}{9} - \frac{3-5h}{54}.$

43. $\frac{1}{2x} - \frac{2}{3x} + \frac{1}{x^2}.$

44. $\frac{3}{ab} - \frac{5}{bc} + \frac{7}{ad}.$

45. $\frac{3}{4y} - \frac{5}{2y^2} + \frac{3}{y^3}.$

46. $\frac{a}{3xy} - \frac{b}{y^2} + \frac{c}{2x^2}.$

47. $\frac{x-1}{2x} + \frac{x-3}{3x}.$

48. $\frac{2a-3}{4a^2} - \frac{5a+3}{6a}.$

49. $\frac{y-2}{3yz} - \frac{2y-3}{2y^2z^2}.$

50. $\frac{5-3u}{2u^2} - \frac{4-2u}{3u}.$

51. $\frac{7-3x}{6} - 1 - \frac{2-4x}{21}.$

52. $-\frac{4a-5}{7} - 2 + \frac{3a+2}{35}.$

53. $\frac{3-2y}{4y^2} + \frac{5+3y}{5y^3}.$

54. $\frac{2-x}{3x^2y} - \frac{4+3y}{2xy^2}.$

55. $\frac{3}{4a^2b^3} - \frac{2}{a} - b.$

56. $5 - x - \frac{3}{2x^2y}.$

40. Mixed expressions

A sum of an integral rational part and of one or more fractions is referred to as a *mixed expression.* If a mixed expression occurs as a factor in a product or as the numerator or denominator of a fraction, it is usually desirable to combine the mixed expression into a single fraction before performing other operations.

ILLUSTRATION 1. We refer to a number such as $5\frac{3}{4}$ as a *mixed number.*

ILLUSTRATION 2. $$(5\tfrac{3}{4})(2\tfrac{3}{7}) = \frac{23}{4}\cdot\frac{17}{7} = \frac{23(17)}{28} = \frac{391}{28}.$$

ILLUSTRATION 3. $3x^2 + \dfrac{5}{x}$ is a mixed expression.

ILLUSTRATION 4. $$\left(2 + \frac{a}{3}\right)\left(3 - \frac{a}{5}\right) = \frac{6+a}{3}\cdot\frac{15-a}{5}$$
$$= \frac{(6+a)(15-a)}{15} = \frac{90 + 9a - a^2}{15}.$$

41. Complex fractions *

A **simple fraction** is one without any fraction in its numerator or denominator. A *complex fraction* is one having one or more fractions in the numerator and denominator.

SUMMARY. *To reduce a complex fraction to a simple fraction:*

1. *Express the numerator and denominator as simple fractions.*

2. *Form the quotient of the simple fractions and reduce the result to lowest terms.*

ILLUSTRATION 1. $$\frac{1+\frac{3}{5}}{2-\frac{4}{3}} = \frac{\frac{5+3}{5}}{\frac{6-4}{3}} = \frac{\frac{8}{5}}{\frac{2}{3}} = \frac{8}{5}\cdot\frac{3}{2} = \frac{12}{5}.$$

Sometimes it may be convenient to reduce a complex fraction to a simple fraction by the *single* operation of *multiplying both numerator and denominator of the complex fraction by the* LCD *of all simple fractions involved.*

* In this chapter, the problems will avoid questions of factoring which will be met in Chapter 5.

ILLUSTRATION 2. To reduce the given complex fraction of Illustration 1 to a simple fraction, we observe that the LCD of 3/5 and 4/3 is 15. Hence, we multiply both numerator and denominator by 15, observing that $15(\frac{3}{5}) = 9$ and $15(\frac{4}{3}) = 20$:

$$\frac{1+\dfrac{3}{5}}{2-\dfrac{4}{3}} = \frac{15+9}{30-20} = \frac{24}{10} = \frac{12}{5}.$$

ILLUSTRATION 3. $$\frac{3a+\dfrac{2}{5b}}{2a-\dfrac{7}{3b}} = \frac{\dfrac{15ab+2}{5b}}{\dfrac{6ab-7}{3b}}$$

$$= \frac{15ab+2}{5b}\cdot\frac{3b}{6ab-7} = \frac{45ab+6}{30ab-35},$$

where we divided out b from numerator and denominator.

ILLUSTRATION 4. $$\frac{\dfrac{2b-5}{3b}}{4-b} = \frac{\dfrac{2b-5}{3b}}{\dfrac{4-b}{1}} = \frac{2b-5}{3b}\cdot\frac{1}{4-b} = \frac{2b-5}{12b-3b^2}.$$

DEFINITION I. *The* **reciprocal** *of a number H is* $\dfrac{1}{H}$.

ILLUSTRATION 5. The reciprocal of 3 is $\frac{1}{3}$.

ILLUSTRATION 6. The reciprocal of $\dfrac{3}{4}$ is $\dfrac{1}{\frac{3}{4}} = \dfrac{1}{1}\cdot\dfrac{4}{3} = \dfrac{4}{3}$.

The reciprocal of $\dfrac{a}{b}$ is

$$\frac{1}{\dfrac{a}{b}} = \frac{1}{1}\cdot\frac{b}{a} = \frac{b}{a}.$$

Thus, the reciprocal of a fraction is the fraction *inverted.*

EXERCISE 15

Express the fraction or product as a simple fraction in lowest terms.

1. $(4 + \frac{3}{5})(5\frac{2}{3})$.
2. $(3 - \frac{2}{3})(4\frac{2}{5})$.
3. $(2\frac{5}{7})(3\frac{2}{3})$.
4. $(b + \frac{3}{4})(b - \frac{2}{5})$.
5. $(\frac{2}{7} - 3b)(\frac{3}{5} - 4b)$.

6. $\dfrac{4+\frac{5}{3}}{1+\frac{2}{5}}$. **7.** $\dfrac{15-\frac{2}{3}}{\frac{1}{3}+\frac{1}{5}}$. **8.** $\dfrac{3-\frac{1}{4}}{2+\frac{1}{4}}$.

9. $\dfrac{2+\frac{3}{8}}{\frac{1}{2}+5}$. **10.** $\dfrac{18}{1-\frac{5}{3}}$. **11.** $\dfrac{2+\frac{9}{11}}{5}$.

12. $\dfrac{\frac{2}{3}-x}{\frac{3}{4}+x}$. **13.** $\dfrac{3a+\frac{2}{5}}{2a-\frac{1}{3}}$. **14.** $\dfrac{5c-\frac{3}{8}}{\frac{3}{4}-2c}$.

15. $\dfrac{\frac{2}{a}-3}{\frac{5}{a}+4}$. **16.** $\dfrac{\frac{4}{b}-3}{5+\frac{3}{b}}$. **17.** $\dfrac{\frac{5}{bc}-3}{4+\frac{3}{bc}}$.

18. $\dfrac{\frac{3}{2x}+\frac{2}{y}}{\frac{2}{x}-\frac{3}{y}}$. **19.** $\dfrac{\frac{3}{2x}-\frac{5}{3y}}{\frac{2}{x}+\frac{3}{4y}}$. **20.** $\dfrac{\frac{5x}{2}+\frac{3}{5x}}{\frac{5x}{3}+\frac{5}{3x}}$.

21. $\dfrac{5-\frac{3}{a^2b}}{7-\frac{4}{2a^2b}}$. **22.** $\dfrac{\frac{3}{a^2}-\frac{5}{2b}}{\frac{2}{a}-\frac{3}{4ab}}$. **23.** $\dfrac{\frac{5}{2x^2}-\frac{3}{y}}{\frac{2}{3y^2}-\frac{1}{2x}}$.

24. $\dfrac{\frac{3}{2a^2h}-\frac{1}{5h^2}}{1-\frac{3}{2ah}}$. **25.** $\dfrac{\frac{5}{2xw}-3}{\frac{4}{2w}-\frac{3}{5x^2w}}$. **26.** $\dfrac{\frac{w^2}{3}+\frac{2}{4x^2}}{\frac{w}{5x}-\frac{3}{4}}$.

27. $\dfrac{5+3b}{2+\frac{3}{a}}$. **28.** $\dfrac{2-3x}{\frac{3}{x}+\frac{2}{y}}$. **29.** $\dfrac{4+3y}{\frac{2}{x}-3}$.

30. $\dfrac{2+\frac{3}{y}}{4y}$. **31.** $\dfrac{\frac{2a}{3}+\frac{5b}{4}}{3x-2}$. **32.** $\dfrac{\frac{2x}{3y}-\frac{y}{7}}{x+3y}$.

Find the reciprocal of the expression, and express the result as a simple fraction in lowest terms.

33. 75. **34.** 17. **35.** $\frac{8}{3}$. **36.** $\frac{5}{7}$. **37.** $-\frac{2}{3}$.

38. $10\frac{2}{3}$. **39.** $12\frac{3}{5}$. **40.** $2\frac{3}{4}$. **41.** $-5\frac{1}{3}$. **42.** $\frac{2}{3}a$.

43. $3b-2a$. **44.** $\dfrac{5x-3}{2x+5}$. **45.** $\dfrac{3a+5h}{2a-3h}$.

46. By writing a fraction, show that *to divide a number N by a number H* means the same as *to multiply N by the reciprocal of H.*

EXERCISE 16

Review of Chapters 1 and 2

Compute each expression, leaving any fraction in lowest terms.

1. $(-3)(-4)(-5)$. **2.** $-(-2)(-5)$. **3.** $-(-3)(-4)(0)$.

4. $\dfrac{(-7)(-3)}{-14}$. **5.** $-\dfrac{(-2)(5)}{(-3)(-6)}$. **6.** $-\dfrac{(-3)(5)}{(-2)(-4)}$.

7. $-7+19-16$. **8.** $3-(-4)-7$. **9.** $-2+(-3)+6$.

Find the absolute value of the expression.

10. -8. **11.** $(-3)(-2)$. **12.** $|-14|$. **13.** $|17|$.

(*a*) *Add the two numbers;* (*b*) *subtract the lower one from the upper number.*

14. $\begin{array}{r} 17 \\ \underline{-5} \end{array}$ **15.** $\begin{array}{r} -23 \\ \underline{-9} \end{array}$ **16.** $\begin{array}{r} -15 \\ \underline{29} \end{array}$ **17.** $\begin{array}{r} 17 \\ \underline{25} \end{array}$

18. Read the expressions $-4 < -2$ and $17 > 0$, and verify their truth by marking the numbers on a real number scale.

Insert the proper sign, $<$ *or* $>$, *between the numbers.*

19. 11 and 19. **20.** -15 and -27. **21.** 0 and -6.

22. Which one of the numbers -15 and 7 is less than the other? Which one is numerically less than the other?

Perform any indicated operation, removing any parentheses, reducing any final fraction to lowest terms, and employing the laws of exponents to simplify expressions.

23. $-(3a-2b-c)$. **24.** $2(3-5a-c)$. **25.** $-a^2(3a^3b-ab^2)$.

26. $5(-3x)$. **27.** $3x^2y^3(2x-4x^2y)$. **28.** $-2x^3y^4(-3x-5y)$.

29. $3(2a-h+k)-2(3a+4h-3k)$.

30. $-x^2(3xy^3-2xy+3)+4y(2x^3y^2-3x^3+5)$.

31. $3a-[2a-3(5-a)]$. **32.** $-(6a-b)-2[3-(2a-4b)]$.

33. $\dfrac{-5}{7}$. **34.** $\dfrac{-15}{35}$. **35.** $\dfrac{4}{7}\cdot\dfrac{14}{15}$. **36.** $\dfrac{3}{5}\div\dfrac{6}{35}$.

37. $17\left(\dfrac{3}{7}\right)$. **38.** $\dfrac{3}{8}\div b$. **39.** $7\div\dfrac{3}{4}$. **40.** $\dfrac{cd^2}{7}\div c^3d^4$.

41. $-2h^3k^2(3hk^4)(-4h^2k)$. **42.** $-4h^2y^3(2hy^2-3h^2y-5h)$.

43. $(3x^2y)^4$. **44.** $(-2a^2x^3)^3$. **45.** $(-5c^2d^3y)^4$.

46. $(-2)^5$. **47.** $\left(\frac{3}{5}\right)^3$. **48.** $\left(-\frac{2}{3}\right)^2$. **49.** $\left(\frac{1}{2}ab\right)^2$. **50.** $\left(\frac{1}{3}c^2d\right)^3$.

51. $\left(\frac{3a}{2x}\right)^2$. **52.** $\left(\frac{c^2x}{4a}\right)^3$. **53.** $\left(\frac{3}{x^3}\right)^4$. **54.** $\left(\frac{-2}{5a^2}\right)^3$.

55. $(2x+3)(2x-7)$. **56.** $(3x^2-7x)(2x-5x^3)$.

57. $(2x-y)(3x-5y)$. **58.** $(2a-3b)^2$.

59. $(3x^2-5x-2x^3+3)(4x+5)$.

60. $\frac{x^4}{x^9}$. **61.** $\frac{a^7}{a^3}$. **62.** $\frac{10ab^3}{25a^2b}$. **63.** $\frac{-6c^3d^2}{24c^5d}$.

64. $(6a^3-19a^2+21a-9) \div (2a-3)$.

65. $(18+4x^3-26x-2x^2) \div (2x-5)$.

66. $\frac{3a^2h^3-5ah^4}{2a^2h^5}$. **67.** $\frac{4y^3+8y^9}{-2y^7}$. **68.** $\frac{4x^3-7x^2}{-2x^4}$.

Express as a single fraction in lowest terms.

69. $\frac{2}{3}-\frac{7}{3}+\frac{10}{3}$. **70.** $\frac{2h-3}{5}-3-\frac{4h-7}{10}$.

71. $\frac{-2a}{3}+2-\frac{4a-7}{12}$. **72.** $\frac{b}{2xy}-\frac{c}{3y^2}+\frac{d}{4x^2}$.

73. $\frac{2-3x}{4x^2y}-\frac{5-3y}{3xy^3}$. **74.** $\frac{2a}{3}-\frac{3}{4a^2b}-\frac{3a-b}{5ab^3}$.

75. $\dfrac{3-\frac{5}{3}}{2+\frac{4}{5}}$. **76.** $\dfrac{\frac{2}{3a}-\frac{5}{2b}}{\frac{3}{a}-\frac{4}{5b}}$. **77.** $\dfrac{\frac{3}{2x^2}-\frac{5}{y}}{\frac{2}{3y^2}-\frac{1}{2x}}$.

78. $\dfrac{3+5a}{2+\frac{3}{a}}$. **79.** $\dfrac{2-3x}{\frac{3}{x}-\frac{5}{y}}$. **80.** $\dfrac{\frac{5}{3}}{2a-3x}$.

81. Find the reciprocal of $\frac{3}{5}$; of $\frac{1}{3}(a-5)$.

CHAPTER 3

DECIMALS AND ELEMENTS OF COMPUTATION

42. Decimal notation

The decimal notation * for writing numbers is called a *place system*, for which the *base* is 10. In this notation, each number is an abbreviation for a sum involving units and powers of ten, written by use of the Hindu-Arabic *digits* or *figures* (0, 1, 2, 3, 4, 5, 6, 7, 8, 9).

ILLUSTRATION 1.
$$\begin{aligned} 3456 &= 3(1000) + 4(100) + 5(10) + 6 \\ &= 3(10^3) + 4(10^2) + 5(10) + 6. \end{aligned}$$

We have 3456 as the sum of 6 *units* or *ones*, 5 *tens*, 4 *hundreds*, and 3 *thousands*.

ILLUSTRATION 2.
$$\begin{aligned} 23.572 &= 2(10) + 3 + \frac{5}{10} + \frac{7}{100} + \frac{2}{1000} \\ &= 2(10) + 3 + \frac{5}{10} + \frac{7}{10^2} + \frac{2}{10^3}. \end{aligned}$$

Counting to the left of the decimal point in a number, the places are named the *units' place* or *ones' place; tens' place; hundreds' place; thousands' place; ten thousands' place;* etc. Counting to the right of the decimal point, we have the *tenths' place; hundredths' place; thousandths' place;* etc. The name of any place is the value of a unit located there. These values should be remembered in terms of powers of ten.

ILLUSTRATION 3. $1000 = 10^3$; $\quad .001 = \frac{1}{1000} = \frac{1}{10^3}$; $\quad .01 = \frac{1}{10^2}$.

ILLUSTRATION 4. We read .5073 as "*point*, 5, *oh*, 7, 3," or as "5073 *ten-thousandths*."

* In this chapter, unless otherwise specified, any number referred to will be positive.

In this chapter we will think of each number in its decimal form. The part to the right of the decimal point is called the **decimal part** of the number. We consider each number as having a decimal part even when the decimal part is zero and the number is then an integer. The **decimal places** of a number are those places to the right of the decimal point in which digits are actually written in the number.

ILLUSTRATION 5. 23.507 has three decimal places and its decimal part is .507.

In any number, observing its digits from left to right, we may visualize an endless sequence of zeros at the right of the last digit not zero, if there is such a last digit. A number of this character is called a **terminating decimal.** An **endless decimal** is one in which, however far we proceed to the right, we never reach a digit beyond which all digits are zeros. Hereafter, unless otherwise stated, any number mentioned will be a terminating decimal.

ILLUSTRATION 6. 35.675, or $35.67500 \cdots$, is a terminating decimal. The familiar number $\pi = 3.14159 \cdots$ is an endless but not a repeating decimal. The fraction $\frac{1}{3}$ is equal to the endless repeating decimal $.333 \cdots$.

If a unit in any place in the decimal notation is *multiplied* by 10, we obtain a unit in the next place to the *left.* If we *divide* a unit in any place by 10, we obtain a unit in the next place to the *right.* The preceding remarks justify the following convenient rules.

I. *To multiply a number by* 10, *move the decimal point one place to the right in the number.*

II. *To divide a number by* 10, *move the decimal point one place to the left in the number.*

ILLUSTRATION 7. $\quad 10(315.67) = 3156.7. \qquad \dfrac{315.67}{10} = 31.567.$

To divide by 1000, which is 10(10)(10), we move the decimal point three places to the left: $\quad 21.327 \div 1000 = .021327.$

43. Addition of decimals

In finding a sum of decimals, after they have been arranged with the decimal points in a column, it is desirable to add once going upward and then downward to check in each column.

ILLUSTRATION 1. To find the sum of 31.457, 2.6, 3.15, and 101.41, we annex zeros to extend each number to the 3d decimal place, and then add.

$$\begin{array}{rr} & 31.457 \\ & 2.600 \\ & 3.150 \\ & \underline{101.410} \\ \textit{Sum} = & 138.617 \end{array}$$

EXAMPLE 1. Add:

$$1.573 + 3.671 - 1.157 - 4.321 + 10.319.$$

SOLUTION. The sum of the positive terms is 15.563; of the negative terms is $-$ 5.478.

2. Sum of all terms is 10.085: $\left\{ \begin{array}{r} 15.563 \\ \underline{-\ 5.478} \end{array} \right\}$ *sum* = 10.085.

EXERCISE 17

Write the number in decimal form.

1. 3 and 25 hundredths.
2. Point, oh, 1, 5, 3, 9, oh, 4.
3. 10^5.
4. 10^7.
5. $\frac{1}{10^4}$.
6. $\frac{1}{10^5}$.
7. $\frac{1}{10^7}$.

Write each number as a sum involving powers of 10, *with one term corresponding to each digit (not zero) of the number.*

8. 567.
9. 3149.
10. 16,342.
11. .319.
12. 27.0457.

Write the number in decimal notation equal to the sum.

13. $5(100) + 3(10) + 6 + \frac{4}{10} + \frac{3}{100} + \frac{7}{1000}$.

14. $2(1000) + 4(10) + 3 + \frac{1}{10} + \frac{8}{1000} + \frac{9}{10,000}$.

15. $5(10^3) + 7(10^2) + 3(10) + 5 + \frac{3}{10} + \frac{5}{10^2}$.

Compute the indicated sum.

16. $2.057 + 3.11 + 4.985 + 3.05 + 1.5 + 2.177 + .459$.
17. $3.193 + .098 + 1.567 + 2.457 + 3.167 + 2.13 + 1.5072$.
18. $21.675 - 14.521$.
19. $.0938 - .0257$.
20. $1.721 - 2.468$.

(*a*) *Add the numbers.* (*b*) *Subtract the lower number from the upper one.*

21. $\begin{array}{r} 5.26 \\ \underline{1.38} \end{array}$
22. $\begin{array}{r} .357 \\ \underline{.2983} \end{array}$
23. $\begin{array}{r} -\ 43.8468 \\ \underline{-\ 59.923} \end{array}$
24. $\begin{array}{r} .02438 \\ \underline{-\ .5729} \end{array}$

25. Compute $2.156 - 3.149 + 4.183 + 2.147 - 4.159$.

26. Compute $13.083 + 2.148 - 41.397 - 2.453 + 12.938$.

Perform the multiplication or division mentally.

27. $10(.532)$. **28.** $1000(1.0219)$. **29.** $10^4(32.653)$. **30.** $100(.00415)$.

31. $\dfrac{.0317}{100}$. **32.** $\dfrac{13.257}{1000}$. **33.** $\dfrac{57.38}{10}$. **34.** $\dfrac{.0498}{1000}$.

44. Multiplication of decimals

Any number whose decimal part is not zero can be written as a fraction whose numerator is an integer and denominator is a power of 10. The exponent of 10 in the denominator can be taken equal to the number of decimal places in the given number.

ILLUSTRATION 1. $$1.21(.205) = \frac{121}{100} \cdot \frac{205}{1000}$$

$$= \frac{121}{10^2} \cdot \frac{205}{10^3} = \frac{(121)(205)}{10^{2+3}} = \frac{24{,}805}{10^5} = .24805.$$

We conclude that the *five* decimal places in the result are a consequence of the law of exponents for multiplication, because the factors had *two* and *three* decimal places. This result is a special case of the following rule.

SUMMARY. *To multiply two decimals:*

1. *Find the digits of the product by multiplying the factors with their decimal points disregarded (or even removed).*

2. *Add the numbers of decimal places in the factors to find the number of decimal places in the product, and insert the decimal point.*

ILLUSTRATION 2. To multiply $.0238 \times 112.75$, we find the digits of the product, at the right, and then point off $(2 + 4)$ or 6 decimal places to obtain 2.683450. Notice that the *final zero had to be written and counted* in fixing the decimal point. In stating the final result, we could then omit the zero.

$$\begin{array}{r} 11275 \\ (\times)\ 238 \\ \hline 90200 \\ 33825 \\ 22550 \\ \hline 2683450 \end{array}$$

The digits in the product of two decimals depend only on the digits of the factors. If the decimal points are moved in the factors, this only alters the position of the decimal point in the result.

ILLUSTRATION 3. In Illustration 2, $.0238(112.75) = 2.683450.$

Hence, $2.38(1127.5) = 2683.450.$

EXERCISE 18

Perform the following multiplications.

1. 3.51(1.4).	**2.** .46(.107).	**3.** 13(.461).
4. .0142(3.6).	**5.** 21.38(.024).	**6.** 156.1(1.38).
7. 398.4(.0342).	**8.** .00175(.2147).	**9.** .0346(.00157).
10. 85.2(1.356).	**11.** 9.137(.2346).	**12.** 74.308(.00259).

45. Significant digits

In any number N, let us read its digits from left to right. Then, by definition, the *significant digits* or *figures* of N are its digits, in sequence, starting with the first one not zero and ending with the last one definitely specified. Notice that this definition does not involve any reference to the position of the decimal point in N. Usually we do not mention *final zeros* at the right in referring to the significant digits of N, except when it is an approximate value.

ILLUSTRATION 1. The significant digits of 410.58 or of .0041058 are (4, 1, 0, 5, 8).

46. Approximate values

If T is the *true* value and A is an *approximate* value of a quantity, we agree to call $A - T$ the *error* of A.

ILLUSTRATION 1. If $T = 35.62$, and if $A = 35.60$ is an approximation to T, then the error of A is $35.60 - 35.62$, or $-.02$.

The significant digits in an approximate value A should indicate the maximum possible error of A. This error is understood to be *at most one half of a unit in the last significant place in* A, or, which is the same, *not more than* 5 *units in the next place to the right.*

ILLUSTRATION 2. If a surveyor measures a distance as 256.8 yards, he should mean that the error is at most .05 yard and that the true result lies between 256.75 and 256.85, since the error (plus or minus) might be $\pm .05$.

In referring to the significant digits of an *approximate* value A, *it is essential to mention all final zeros designated in* A.

ILLUSTRATION 3. To state that a measured weight is 35.60 pounds should mean that the true weight differs from 35.60 pounds by at most .005 pound. To state that the weight is 35.6 pounds should mean that the true weight

differs from this by at most .05 pound. Thus, there is a great distinction between 35.6 and 35.60 as *approximate values* although there is no difference between 35.6 and 35.60 as *abstract numbers.*

47. Rounding off a number

In referring to a *place* in a number, we shall mean any place where a significant digit stands. In referring to a *decimal place*, the word *decimal* will be explicitly used.

To round off N to k figures, or to write a k-place approximation for N, means to write an approximate value with k significant digits so that the error of this value is not more than one half of a unit in the kth place, or 5 units in the first neglected place. This condition on the approximate value of N leads us to the following method.

SUMMARY. *To round off a number N to k places, drop off the part of N beyond the kth place and then proceed as follows:*

1. *If the omitted part of N is less than* 5 *units in the* $(k + 1)$th *place, leave the digit in the* kth *place unchanged.*

2. *If the omitted part of N is more than* 5 *units in the* $(k + 1)$th *place, increase the digit in the* kth *place by* 1.

3.* *If the omitted part of N is exactly equal to* 5 *units in the* $(k + 1)$th *place, increase the digit in the* kth *place or leave it unchanged, with the object of making the final choice an even digit.*

ILLUSTRATION 1. The seven-place approximation to π is 3.141593. On rounding off to *five places* (or *four decimal places*) we obtain 3.1416. We changed 5 to 6 because .000093 > .00005. On rounding off π to three places we obtain 3.14.

ILLUSTRATION 2. In rounding off 315.475 to five figures, with equal justification we could specify either 315.47 or 315.48 as the result. In accordance with Item 3 of the Summary, we choose 315.48.

48. A notation for large numbers

For abbreviation, or to indicate how many digits in a large number are significant, it is sometimes convenient to write a number N as the product of *an integral power of* 10 *and a number equal to or greater than* 1 *but less than* 10, with as many significant digits as are justified by the data.

* Item 3 could be replaced by various similar and equally justified agreements.

ILLUSTRATION 1. If 5,630,000 is an approximate value, its appearance fails to show how many zeros are significant. If just five digits are significant we write $5.6300(10^6)$, and, if just three are significant, $5.63(10^6)$.

49. Accuracy of computation

By illustrations, we can verify that the following rules do not *underestimate* the accuracy of computation with approximate values. On the other hand, we must admit that the rules sometimes *overestimate* the accuracy. However, we shall assume that a result obtained by these rules will have a negligible error in the last significant place which is specified.

I. *In adding approximate values, round off the result in the first place where the last significant digit of any given value is found.*

II. *In multiplying or dividing approximate values, round off the result to the smallest number of significant figures found in any given value.*

ILLUSTRATION 1. Let $a = 35.64$, $b = 342.72$, and $c = .03147$ be approximate values. Then, $a + b + c$ is not reliable beyond the *second* decimal place because both a and b are subject to an unknown error which may be as large as 5 units in the third decimal place. Hence, we write

$$a + b + c = 378.39147 = 378.39, \textit{ approximately.}$$

ILLUSTRATION 2. If $x = 31.27$ and $y = .021$ are approximate values, then, by Rule II, we take $xy = .66$, because y has only two significant digits:

$$xy = 31.27(.021) = .65667 = .66, \textit{ approximately.}$$

ILLUSTRATION 3. If a surveyor measures a rectangular field as 385.6′ by 432.4′, it would give an unjustified appearance of accuracy to write that the area is $(385.6)(432.4) = 166{,}733.44$ square feet. For, an error of .05 foot in either dimension would cause an error of about 20 square feet in the area. By Rule II, a reasonably justified result would be that the area is 166,700 square feet, to the nearest 100 square feet, or $1.667(10^5)$ square feet.

In problems where approximate values enter, or where approximate results are obtained from exact data, the results should be rounded off so as to avoid giving a false appearance of accuracy. No hard and fast rules for such rounding off should be adopted, and the final decision as to the accuracy of a result should be made only after a careful examination of the details of the solution.

EXERCISE 19

Round off, first to five and then to three significant digits.

1. 15.32573. **2.** .00132146. **3.** .3148638. **4.** 5.62653.
5. 195.635. **6.** .00128558. **7.** .0345645. **8.** 392.462.

Tell between what two values the exact length of a line lies if its measured length in feet is as follows.

9. 567. **10.** 567.0. **11.** 567.4. **12.** 35.18.

Assuming that the numbers represent approximate data, find their sum and product and state the results without false accuracy.*

13. 31.52 and .0186. **14.** .023424 and 1.14. **15.** .0047(11.3987126).

HINT for Problem 15. In proceeding to multiply or divide approximate values, there is no advantage in keeping *many* significant digits in *one* value when other values have relatively *few* significant digits. A conservative rule is that, before multiplying or dividing, we may *round off any given value to two more significant digits than appear in the least accurate of the given values.*†

Write the number in ordinary decimal form.

16. 100(3.856). **17.** 27.38(10^2). **18.** 1.935(10^4). **19.** 2.056(10^6).

Write as the product of a power of 10 *and a number between* 1 *and* 10.

20. 38.075. **21.** 675.38. **22.** 153,720,000. **23.** 45,726.

Given that the number is an approximate value, write it as the product of an integral power of 10 *and a number between* 1 *and* 10, *assuming, first, that there are five significant digits and, second, that there are three significant digits.*

24. 9,330,000. **25.** 453,120. **26.** 23,523,416. **27.** 72,200,000.

In the following problems, state each result without false accuracy.

28. The measured dimensions of a rectangular field are 469.2 feet and 57.3 feet. Find the perimeter (sum of lengths of sides) and area of the field.

29. The measured dimensions of a rectangular box are 20.4 feet, 16.5 feet, and 7.8 feet. Find the volume of the box in cubic feet.

30. Given that one cubic foot contains approximately 7.5 gallons, how many gallons are contained by 2.576 cubic feet?

* In this book, unless otherwise stated, the numerical data in any problem should be assumed to be exact. Results obtained from exact data may sometimes be rounded off.

† See Note 3 in the Appendix for a convenient abridged method for multiplyin two numbers with many significant digits.

50. Division of decimals

When one decimal is divided by another, the division process sometimes gives a zero remainder if the work is carried to a sufficient number of decimal places. Usually, however, we may expect a remainder not zero however far we continue the division. We can always arrange the details so that the actual work amounts to division by an *integer.* This arrangement is useful in locating the decimal point in the result.

ILLUSTRATION 1. To compute $372.173 \div 1.25$ with accuracy to two decimal places, we first indicate the division as a fraction, and then multiply its numerator and denominator by 100, to obtain *an integer as the divisor:*

$$\frac{372.173}{1.25} = \frac{372.173(100)}{1.25(100)} = \frac{37{,}217.3}{125}. \tag{1}$$

$$\begin{array}{r l}
 & 2\,97.738\ (\textit{Quotient}) \\
1.25_\wedge & \overline{)\,372.17_\wedge 300}\ (\textit{Dividend}) \\
 & \underline{250} \\
 & 122\,1 \\
 & \underline{112\,5} \\
 & 9\,67 \\
 & \underline{8\,75} \\
 & 92\,3 \\
 & \underline{87\,5} \\
 & 4\,80 \\
 & \underline{3\,75} \\
 & 1\,050 \\
 & \underline{1\,000} \\
 & 50
\end{array}$$

At the right, we insert the original decimal points in dividend and divisor and mark with "$_\wedge$" the new locations of the decimal points observed in the final fraction in (1). The multiplication by 100 in (1) is equivalent to the action of moving the decimal point *two places to the right* in both dividend and divisor. In the process of division, the *integral part* of the quotient *ends* when we begin using digits of the dividend at *the right of the new decimal point,* "$_\wedge$." If we place each digit of the quotient directly above the last digit being used at that stage from the dividend, the decimal point in the quotient occurs exactly above the altered decimal point, $_\wedge$, in the dividend. We find the quotient to three decimal places, 297.738, and then round it off to 297.74, which we say is correct to two decimal places. To check, we compute

$$1.25(297.74) = 372.175.$$

We do not obtain *exactly* 372.173 because 297.74 is not the *exact* quotient.

Note 1. In any division, estimate the result before dividing, to check the location of the decimal point in the quotient. Thus, in Illustration 1, a convenient estimate would be $375 \div 1.25$, or 300. Then we are sure that the actual answer should have three places to the left of the decimal point.

Note 2. In dividing approximate values, obtain the quotient to *one more figure* than is specified as reliable by Rule II, Section 49. Then, round off the quotient in the preceding place.

Any terminating decimal can be expressed as a fraction, which then may be reduced to lowest terms. Conversely, we can express any common fraction as a decimal by carrying out the division indicated by the fraction, to obtain either a terminating or an endless decimal.

ILLUSTRATION 2. On carrying out the division, we obtain $\frac{7}{8} = .875$, exactly.

ILLUSTRATION 3. $$3.125 = \frac{3125}{1000} = \frac{25}{8}.$$

ILLUSTRATION 4. To express $\frac{11}{24}$ as a decimal we divide, at the right. After reaching .458 in the quotient, we meet 8 each time as the only digit in the remainder. Hence, 3 will be obtained endlessly in the quotient. The result is the endless repeating decimal $.458\dot{3}$, where the dot above 3 means that the digit repeats endlessly.

```
        .45833 · · ·
24 ) 11.00000
      9 6
      1 40
      1 20
        200
        192
          80
          72
          80 etc.
```

EXERCISE 20

(*a*) *Write each expression as a fraction, and then alter it to make the divisor an integer.* (*b*) *Divide until the remainder is zero.*

1. $3.562 \div 2.6$. **2.** $2.849 \div .74$. **3.** $140.14 \div 2.45$.

Obtain the result of the division correct to three decimal places.

4. $381.32 \div 58$. **5.** $.083172 \div .316$. **6.** $.5734 \div 12.8$.

Assume that the numbers in the following problems are approximate values. Carry out the division and state the result without false accuracy, according to Rule II, *Section* 49.

7. $573.2 \div 3.83$. **8.** $19.438 \div 2.21$. **9.** $.09734 \div 3.265$.

10. $98.3 \div 21.473$. **11.** $.003972 \div .0139$. **12.** $.01793 \div .5634$.

Change the decimal to a fraction in lowest terms.

13. 2.75. **14.** .0125. **15.** 2.375. **16.** .0175. **17.** .0325.

Obtain the decimal equal to the given fraction. If the decimal repeats endlessly, carry the division far enough to justify this conclusion.

18. $\frac{3}{4}$. **19.** $\frac{5}{8}$. **20.** $\frac{5}{12}$. **21.** $\frac{5}{33}$. **22.** $\frac{3}{16}$. **23.** $\frac{7}{16}$.

CHAPTER 4

LINEAR EQUATIONS IN ONE UNKNOWN

51. Terminology about equations

An *equation* is a statement that two number expressions are equal. The two expressions are called the *sides* or *members* of the equation. An equation in which the members are equal for all permissible values of the letters involved is called an **identical equation,** or, for short, an **identity.** An equation whose members are *not* equal for all permissible values of the letters is called a **conditional equation.**

ILLUSTRATION 1. *In the following equation, by carrying out the multiplication on the left-hand side, we verify that the product is the same as the* right-hand side. This is true *regardless* of the values of a and b. Hence the equation is an *identity.*

$$(a + 2b)(a + 3b) = a^2 + 5ab + 6b^2.$$

ILLUSTRATION 2. The equation $x - 2 = 0$ is a *conditional* equation because the two members are equal only when $x = 2$.

The word "*equation*" by itself will be used in referring to both identities and conditional equations, except where such usage would cause confusion. Usually, however, the word "*equation*" refers to a *conditional* equation. At times, to emphasize that some equation is an identity, we shall use "$\equiv$" instead of "$=$" between the members.

A conditional equation may be thought of as presenting a question: the equation asks for the values which certain letters should have in order to make the two members equal. These letters, whose values are requested, are called *unknowns.* Some of the letters in an equation may represent known numbers.

ILLUSTRATION 3. $x^2 + 3x - 5 = 0$ is an equation in the unknown x.

52. Solution of an equation

An equation is said to be *satisfied* by a set of values of the unknowns if the equation becomes an identity when these values are substituted for the unknowns. A **solution** of an equation is a set of values of the unknowns which satisfies the equation. A solution of an equation in a single unknown is also called a **root** of the equation. *To solve* an equation in a single unknown means to find all the solutions of the equation.

ILLUSTRATION 1. 4 is a root of the equation $2x - 3 = 5$, because when $x = 4$ the equation becomes $[2(4) - 3] = 5$, which is true.

53. Equivalent equations

Two equations are said to be **equivalent** if they have the same solutions.

EXAMPLE 1. Solve:
$$3 - 3x = -7 - 5x. \tag{1}$$

SOLUTION. 1. Add $5x$ to both members:
$$3 - 3x + 5x = -7 - 5x + 5x, \quad \text{or} \quad 3 + 2x = -7. \tag{2}$$

2. Subtract 3 from both members, or, which is the same, add -3 to both sides:
$$-3 + 3 + 2x = -7 - 3, \quad \text{or} \quad 2x = -10. \tag{3}$$

3. On dividing both sides of (3) by 2 we obtain
$$x = -5, \tag{4}$$
and conclude that this is the only solution of (1).

CHECK. Substitute $x = -5$ in (1).

Left-hand side: $3 - 3(-5) = 3 + 15 = 18.$

Right-hand side: $-7 - 5(-5) = -7 + 25 = 18,$ *which checks.*

Comment. Each equation obtained from (1) was *equivalent* to it and this would justify us in saying that (1) has *just one root,* $x = -5$, without any necessity for the check.

The equivalence of (1), (2), (3), and (4) is a consequence of the following familiar facts: *if equal numbers are added to or subtracted from equal numbers the results are equal; if equal numbers are multiplied or divided by equal numbers the results are equal.*

ILLUSTRATION 1. Any value of x satisfying (1) will also satisfy (2), because we pass from (1) to (2) by adding *equals*, $5x$ and $5x$, to the *equal sides* of (1). And, any value of x satisfying (2) will satisfy (1) because we can pass from (2) *back to* (1) by the *inverse operation* of subtracting $5x$ from both sides of (2). Hence, (1) and (2) are satisfied by the same values of x, and thus are equivalent.

In solving an equation in a single unknown, by use of the following operations we pass from the original equation to progressively simpler equivalent equations, which finally yield the desired roots.

SUMMARY. *Operations on an equation yielding equivalent equations.*

1. *Addition of the same number to both members.*

2. *Subtraction of the same number from both members.*

3. *Multiplication (or division) of both members by the same number, provided that it is not zero and does not involve the unknowns.*

Note 1. We observe that Operation 2 is a special case of Operation 1 because *subtraction* of a number N is equivalent to *addition* of $-N$. The division part of Operation 3 is a special case of the multiplication part, because division by a number N is equivalent to multiplication by $1/N$.

Convenient mechanical processes, and corresponding terminology, grow out of Operations 1, 2, and 3.

A term appearing on both sides of an equation can be **canceled,** *by subtracting the term from both sides.*

ILLUSTRATION 2. Given: $x + 3 = \frac{5}{4} + 3.$

Subtract 3 from both sides: $x = \frac{5}{4}.$

A term can be **transposed** *from one member to the other, with the sign of the term changed, by subtracting it from both members.*

ILLUSTRATION 3. Given: $x + 5 = 7.$

Subtract 5 from both sides, or *transpose* 5:

$$x = 7 - 5, \quad \text{or} \quad x = 2.$$

ILLUSTRATION 4. Given: $x - 4 = 9.$

Transpose -4: $x = 9 + 4, \quad \text{or} \quad x = 13.$

The signs of all terms on both sides may be changed, by multiplying both sides by -1.

ILLUSTRATION 5. Given: $$3x - b = 5 - ax.$$
Change all signs (multiply both sides by -1):
$$-3x + b = -5 + ax.$$

54. Degree of a term

The *degree* of an integral rational term in a certain literal number is the exponent of the power of that number which is a factor of the term. If the term does *not involve* the number, the degree of the term is said to be *zero*. The degree of a term in two or more letters *together* is the *sum of their degrees* in the term. The degree of a polynomial is defined as the degree of its term of *highest* degree.

ILLUSTRATION 1. With x as the literal number involved, the degree of $5x^3$ is 3. The degree of $2x$ is 1 because $x = x^1$. The degree of $(5x^3 - 3x^2 + 2x - 7)$ is 3

ILLUSTRATION 2. The degree of $3x^2y^3$ in x is 2, in y is 3, and in x and y *together* is $(3 + 2)$ or 5.

ILLUSTRATION 3. A polynomial of the *first* degree in x is called a **linear polynomial** in x and is of the form $ax + b$, where a and b do not involve x.

55. Linear equations

An **integral rational equation** is one in which each member is an integral rational polynomial in the unknowns. A *linear* equation, or an equation of the *first degree*, is an integral rational equation in which the terms involving the unknowns are of the first degree.

ILLUSTRATION 1. The equation $3x - 7 = 5$ is a linear equation in x.

SUMMARY. *To solve a linear equation in one unknown:*

1. *If fractions appear, place parentheses around each numerator and clear of fractions by multiplying both members by the* LCD *of the fractions; then, remove parentheses and combine terms.*

2. *Transpose all terms involving the unknown to one member and all other terms to the other member. Combine terms in the unknown, exhibiting it as a factor.*

3. *Divide both sides by the coefficient of the unknown.*

4. *To check the solution, substitute the result in the original equation.*

EXAMPLE 1. Solve: $$\frac{x-4}{3}-\frac{x-3}{2}=\frac{3+x}{10}-2.$$

SOLUTION. 1. The LCD is 30. Hence, multiply both sides by 30, observing that

$$30\cdot\frac{x-4}{3}=10(x-4);\quad 30\cdot\frac{x-3}{2}=15(x-3);\quad 30\cdot\frac{3+x}{10}=3(3+x):$$

$$10(x-4)-15(x-3)=3(3+x)-60;$$
$$10x-40-15x+45=9+3x-60;$$
$$-5x+5=3x-51.$$

2. Subtract $3x$ and 5 from both sides:

$$-5x-3x=-51-5;\quad -8x=-56.$$

3. Divide both sides by -8: $$x=7.$$

CHECK. Substitute $x = 7$ in the original equation.

Left-hand side: $$\frac{7-4}{3}-\frac{7-3}{2}=\frac{3}{3}-\frac{4}{2}=1-2=-1.$$

Right-hand side: $$\frac{3+7}{10}-2=\frac{10}{10}-2=1-2=-1.$$ *This checks.*

In the case of a linear equation in a single unknown x, if the unknown remains in the equation after Step 2 of the standard method of solution is applied, the equation is then of the form $cx = b$, where $c \neq 0$, and b and c do not involve x. On dividing both sides of $cx = b$ by c we obtain $x = b/c$. That is, a linear equation in a single unknown has *just one root.*

Note 1. In directions for solving an equation, in this book, a specification to *add, subtract, multiply,* or *divide* will mean to perform the operation on both sides of the *preceding* equation.

EXERCISE 21

Solve the equation for the literal number in it.

1. $5x-3=x+7.$

2. $3x-6=18-x.$

3. $5-2y=2-3y.$

4. $3-3x=-7-5x.$

5. $2z+5=4(2-z).$

6. $2y-4=1-4y.$

7. $5-5y=5-4y.$

8. $2(4-x)=8-3x.$

9. $8y+3=5y-4.$

10. $7-x=2(1-4x).$

11. $2(7 + x) = 1 - 7x.$

12. $11h - 5 = 8h - 4.$

13. $9 - 7h = 14h - 12.$

14. $7k + 12 = 2k - 7.$

15. $5z - 11 + 3z = z - 3.$

16. $11 - 6w = 34w - 9.$

17. $5x + \frac{3}{4} = 4x + 2.$

18. $4z - \frac{1}{2} = 3z + \frac{5}{2}.$

19. $4x + \frac{1}{6} = 3x - \frac{1}{2}.$

20. $\frac{2}{3} - 3y = -5y - \frac{2}{15}.$

21. $5x - \frac{3}{4} = 3x + \frac{47}{12}.$

22. $5x + \frac{37}{4} = 17x - \frac{11}{4}.$

23. $\frac{3h}{10} - h = \frac{3}{2} - \frac{h}{5}.$

24. $\frac{5x}{3} - 3 = \frac{3x}{4} + \frac{3x}{2}.$

25. $\frac{5y}{6} - 4 = \frac{2y}{3} - \frac{3y}{2}.$

26. $\frac{2x}{15} - \frac{x}{3} = \frac{x}{5} - \frac{6}{5}.$

27. $\frac{3x}{10} - \frac{5}{2} = \frac{x}{6} - \frac{1}{2}.$

28. $\frac{3y}{10} - \frac{y}{3} = \frac{3}{2} - \frac{y}{12}.$

29. $\frac{3x}{2} - \frac{3x}{5} = \frac{3}{5}.$

30. $\frac{4x}{9} - \frac{3}{5} = \frac{5x}{6} - \frac{13}{10}.$

31. $\frac{2y - 7}{3} = \frac{4 + y}{4}.$

32. $\frac{h - 2}{7} = \frac{h - 2}{6}.$

33. $\frac{z - 7}{2} = \frac{1 - z}{10}.$

34. $\frac{3w - 1}{6} - \frac{w + 1}{4} = 0.$

35. $\frac{5 - 2x}{3} = \frac{25}{12} - \frac{5x - 3}{4}.$

36. $\frac{3 - 2x}{3} = \frac{9}{5} - \frac{x - 3}{5}.$

37. $\frac{3 - x}{6} = \frac{5}{6} - \frac{x - 2}{2}.$

38. $\frac{37}{10} + \frac{2x + 7}{5} = -\frac{3x - 5}{4}.$

39. $\frac{4x - 11}{6} = \frac{x - 5}{2} + \frac{7}{2}.$

40. $\frac{3h + 5}{6} - \frac{h - 2}{3} = 2.$

41. $\frac{3x - 2}{4} + \frac{10x - 8}{12} = \frac{13}{4} + \frac{x - 2}{3}.$

42. $\frac{x - 5}{3} + \frac{11x - 3}{15} = 3 + \frac{2x - 1}{5}.$

43. $.26 - z = .98 - 3z.$

44. $3x - .55 = .33 - 5x.$

45. $.26x - .2 = .53x - .38.$

46. $2.3x - 2.4 = 1.6 - 1.7x.$

47. $2.5x - 3.7 = 13.5 - 1.8x.$

48. $.21x - .46 = .79 + .96x.$

49. $.19x - .358 = .032 - .07x.$

50. $4.088 + .03x = 3x - .07.$

51. $2.035 - .637x = -.212x - .215.$

52. $3(5x + 2) - 12 = 25(2x + 1) - 3.$

53. $2y - 1 - 10(y + 1) = 2(2 - 3y) - 1.$

54. $8z - 2(3z - 1) = 7z - 3(z - 1).$

55. $2x - 6(x + 1) = 1 - 3(3x - 1).$

56. $8(w + 2) - 5(2w - 1) = 5(w - 2) + 3.$

57. $9\left(\frac{x}{3} - 1\right) + 5\left(\frac{4x}{5} + 3\right) - 3\left(2x + \frac{3}{2}\right) = 0.$

58. $4r - 9 = 7(2 - r) - 6(r + 1).$

Solve for P, or A, correct to two decimal places, by first clearing of fractions.

59. $300 = P[1 + \frac{5}{6}(.07)].$

60. $250 = A[1 - \frac{7}{8}(.05)].$

61. $500 = A[1 - \frac{7}{12}(.05)].$

62. $750 = P[1 + \frac{19}{12}(.07)].$

56. Simple factoring of polynomials

If all terms of a polynomial contain a common factor, we may express the polynomial as a product of this factor and a second factor. The second factor is the sum obtained by dividing each term of the polynomial by the *common factor* of the terms.

ILLUSTRATION 1.

$$3x + ax + bx = x(3 + a + b).$$

$$3xy^2 + 2xy + 4xy^3 = xy(3y + 2 + 4y^2).$$

At present, in solving equations in an unknown x, we will be interested in factoring polynomials only where x is a common factor of the terms. If we express such a polynomial as the product of x and another factor, we then refer to this factor as the *coefficient of* x.

ILLUSTRATION 2. On factoring, we obtain

$$2x + 4ax + 2cx = 2x(1 + 2a + c). \tag{1}$$

In (1), we say that the coefficient of x is $2(1 + 2a + c)$.

57. Constants and variables

In a given problem, a *constant* is a number symbol whose value does not change during the discussion involved. A *variable* is a number symbol which may take on different values. Any explicit number, such as 7, automatically is a constant wherever it appears.

ILLUSTRATION 1. The volume V of a sphere is given by the formula $V = \frac{4}{3}\pi r^3$ where r is the radius. In considering all possible spheres, r and V are variables but π is a constant, approximately 3.1416.

58. Literal equations

Sometimes an equation in an unknown x may involve other literal numbers besides x. In such a case, during the process of solution for x, we assume that the other literal numbers are constants. The summary of Section 55 still specifies our method of solution.

EXAMPLE 1. Solve for x: $$3bx - 2 = 2cx + a.$$

SOLUTION. 1. Transpose -2 and $2cx$:
$$3bx - 2cx = 2 + a. \tag{1}$$

2. Factor on the left in (1):
$$x(3b - 2c) = 2 + a. \tag{2}$$

3. Divide both sides of (2) by $(3b - 2c)$, the coefficient of x:
$$x = \frac{2 + a}{3b - 2c}. \tag{3}$$

EXAMPLE 2. Solve for x: $$\frac{2x}{ab} - \frac{3}{a} = \frac{x}{2a}.$$

SOLUTION. 1. The LCD is $2ab$. Hence, multiply both sides by $2ab$, noticing that
$$2ab\left(\frac{2x}{ab}\right) = 4x; \qquad 2ab\left(\frac{3}{a}\right) = 6b; \qquad 2ab\left(\frac{x}{2a}\right) = bx.$$

We obtain $$4x - 6b = bx.$$

2. Transpose terms, and solve for x:
$$4x - bx = 6b; \qquad x(4 - b) = 6b; \qquad x = \frac{6b}{4 - b}.$$

EXAMPLE 3. Solve for x: $$\frac{3}{2x} - \frac{7}{3x} = -\frac{5}{4}. \tag{4}$$

SOLUTION. The LCD is $12x$. Multiply both sides by $12x$:
$$18 - 28 = -15x; \qquad -10 = -15x; \qquad x = \tfrac{2}{3}. \tag{5}$$

CHECK. Substitute $x = \frac{2}{3}$ in (4):

Left-hand side: $$\frac{3}{2(\frac{2}{3})} - \frac{7}{3(\frac{2}{3})} = \frac{9}{4} - \frac{7}{2} = -\frac{5}{4};$$

Right-hand side: $$-\frac{5}{4}.$$ *This checks.*

Comment. In this chapter, the unknown will occur in a denominator only under the most simple conditions. In solving (4), an incorrect choice of the LCD, containing an unnecessary factor, might have prevented the equations in (5) from being linear in x.

59. Formulas

Frequently, the data in a problem come to us as the values of a set of variables, which we represent as literal numbers. Sometimes we are able to write a mathematical expression for one of the variables in terms of the others. The resulting equation is referred to then as a *formula* for the first variable. An algebraic formula is one involving only the operations of algebra.

ILLUSTRATION 1. The Fahrenheit reading, F, and centigrade reading, C, in degrees for a given temperature are related by the equation $5F = 9C + 160$. On solving for F, we obtain a formula for F in terms of C:

$$F = \tfrac{9}{5}C + 32. \tag{1}$$

To find F corresponding to 36° centigrade, substitute $C = 36$ in (1):

$$F = \tfrac{9}{5}(36) + 32 = 64.8 + 32 = 96.8°.$$

EXERCISE 22

Solve for x, or y, or z, whichever appears. Reduce any final fraction to lowest terms.

1. $bx = 3 + c.$ **2.** $16x - 4 = h.$ **3.** $cx - 5a = 3h.$

4. $ay - by = 5.$ **5.** $3x = ax + 2b.$ **6.** $2z = bz - a.$

7. $2ay - 5c = 3by + 4a.$ **8.** $7x - d = 5ax + 8.$

9. $ax - 3ax = 5c - bx.$ **10.** $2dx - 3 = d^2x + b.$

11. $\dfrac{2x}{b} = a.$ **12.** $3x = \dfrac{c}{b}.$ **13.** $\dfrac{2y}{c} = d.$

14. $\dfrac{ax}{b} - \dfrac{c}{d} = 0.$ **15.** $\dfrac{2b}{c} - \dfrac{3x}{a} = 0.$ **16.** $\dfrac{a^2x}{3} = 2a^3.$

17. $\dfrac{hy}{2k} - \dfrac{h^2}{3k^2} = 0.$ **18.** $\dfrac{b^2z}{2c} - \dfrac{a}{b} = 0.$ **19.** $\dfrac{A}{4} = \dfrac{3x}{BC}.$

20. $\dfrac{x}{b} - 2x = 3.$ **21.** $\dfrac{3}{a} - \dfrac{1}{b} = \dfrac{4}{x}.$ **22.** $\dfrac{5}{x} - \dfrac{3}{a} = \dfrac{2}{3c}.$

23. $\dfrac{ax}{b} + 2c = \dfrac{dx}{a} + 2.$ **24.** $\dfrac{cx}{3} - \dfrac{a}{2c} = \dfrac{2x}{c^2} - 1.$

25. $3a(2x - 3b) - 5(cx - 3) = 2b$. **26.** $4c(ax - bc^3) - 2a(bx + a^2) = 0$.

27. $\dfrac{4}{5x} - \dfrac{2}{3x} = \dfrac{1}{15}$. **28.** $\dfrac{5}{3x} = -\dfrac{29}{24} - \dfrac{3}{4x}$.

29. $\dfrac{1}{10x} = \dfrac{7}{12} + \dfrac{2}{15x}$. **30.** $\dfrac{z - a}{5bz} = \dfrac{10z + a}{15az}$.

31. In the Fahrenheit-centigrade equation, $5F = 9C + 160$, solve for C in terms of F. Then, use the resulting formula to find the centigrade temperature correct to tenths of a degree corresponding to the following Fahrenheit temperatures: (*a*) 32°; (*b*) 212°; (*c*) 80°; (*d*) 50°.

32. Let an object be shot vertically upward from the surface of the earth with an initial velocity of v feet per second, and let us neglect air resistance and other disturbing features. Then, it is proved in physics that $s = vt - \frac{1}{2}gt^2$, where s feet is the height of the object above the surface at the end of t seconds and $g = 32$, approximately. (*a*) Find s if $v = 100$ and $t = 6$. (*b*) Solve for v to obtain a formula for v in terms of s and t. (*c*) From (*b*), compute the velocity with which the object must be shot to attain a height of 1000 feet in 5 seconds.

33. Solve $f = ma$ for a. **34.** Solve $s = k + vt$ for v.

35. Solve $l = a + (n - 1)d$ for a; for n; for d.

36. Solve $l = \dfrac{l - M}{Mt}$ for M; for t; for l.

37. Solve $S = \dfrac{ar^n - a}{r - 1}$ for a.

38. Solve $S = \dfrac{rl - a}{r - 1}$ for l; for a.

Each of the following problems states a rule for computing values of a certain variable in terms of others. State the rule by means of a formula.

39. The average, A, of three numbers M, N, and P is one third of their sum.

40. The cost of electricity for a house in a certain city is 7¢ per kilowatt-hour for the first 10 kilowatt-hours, 5¢ per kilowatt-hour for the next 20 kilowatt-hours, and $2\frac{1}{2}$¢ per kilowatt-hour for all over 30 kilowatt-hours. Write an expression for the total cost, C, of (*a*) 50 kilowatt-hours; (*b*) n kilowatt-hours where $n > 30$.

41. On any order for more than 200 units of a certain manufactured product, the cost is 15¢ per unit for 200 units and 12¢ per unit for the remainder of the order. Write a formula for the cost, C, in dollars, of n units if $n > 200$.

60. Algebraic translation

In applying equations in the solution of problems stated in words, we translate word descriptions into algebraic expressions.

ILLUSTRATION 1. If x is the length of one side of a rectangle and if the other dimension is 3 units less than twice as long, then the other dimension is $(2x - 3)$; the perimeter (sum of lengths of sides) is

$$2x + 2(2x - 3), \quad \text{or} \quad 6x - 6,$$

and the area is $x(2x - 3)$.

ILLUSTRATION 2. If x, y, and z are, respectively, the *units'*, *tens'*, and *hundreds'* digits of a positive integer with three digits, the value of the integer is $x + 10y + 100z$.

SUMMARY. *To solve an applied problem by use of equations:*

1. *Introduce one or more letters to represent the unknowns and give a description of each one in words.*

2. *Translate the given facts into one or more equations involving the unknowns, and solve for their values.*

3. *Check the solution by substituting the results in the written statement of the problem.*

EXAMPLE 1. \$350 is to be divided between Jones and Smith so that Jones will receive \$25 more than Smith. How much does each receive?

SOLUTION. 1. Let x be the number of dollars which Smith receives. Then, Jones receives $(x + 25)$ dollars.

2. The sum of the amounts received is \$350, or

$$x + (x + 25) = 350. \tag{1}$$

On solving (1), we obtain $x = 162.50$. Hence, Smith receives \$162.50 and Jones receives \$162.50 + \$25 or \$187.50. These results check.

EXAMPLE 2. Find two consecutive *even* integers such that the square of the larger is 44 greater than the square of the smaller integer.

SOLUTION. 1. Let x represent the smaller integer. Then, the larger integer is $x + 2$. Their squares are x^2 and $(x + 2)^2$.

2. From the data, $$(x + 2)^2 - x^2 = 44. \tag{2}$$

Expanding: $$x^2 + 4x + 4 - x^2 = 44;$$

$$4x = 40; \qquad x = 10.$$

3. The integers are 10 and 12.

CHECK. $10^2 = 100$; $12^2 = 144$; $144 - 100 = 44$.

EXAMPLE 3. How long will it take Jones and Smith, working together, to plow a field which Jones can plow alone in 5 days and Smith, alone, in 8 days?

SOLUTION. 1. Let x days be the time required by Jones and Smith, working together.

2. In 1 day, Jones can plow $\frac{1}{5}$ and Smith $\frac{1}{8}$ of the field. Hence, in x days Jones can plow $\frac{x}{5}$ and Smith can plow $\frac{x}{8}$ of the field.

3. Since the *whole* field is plowed in x days, the sum of the fractional parts plowed by the men in x days is 1:

$$\frac{x}{5} + \frac{x}{8} = 1; \quad 8x + 5x = 40; \quad x = 3\tfrac{1}{13} \text{ days.}$$

EXERCISE 23

Solve by use of an equation in just one unknown.

1. A line 68 inches long is divided into two parts where one is 3 inches longer than the other. Find their lengths.

2. A rope 36 inches long is cut into two pieces such that one part is 4 inches less than twice as long as the other part. Find the lengths of the parts.

3. Find the dimensions of a rectangle whose perimeter is 55 feet, if the altitude is $\frac{2}{9}$ of the base.

4. One dimension of a rectangle is $\frac{2}{3}$ the other. Find the dimensions of the rectangle if its perimeter becomes 130 feet when each dimension is increased by 5 feet.

5. What number should be subtracted from the numerator of $\frac{21}{29}$ to cause the fraction to equal $\frac{3}{5}$?

6. Find two consecutive positive integers whose squares differ by 27.

7. Find three consecutive integers whose sum is 48.

8. Find the angles of a triangle where one angle is three times a second angle and six times the third angle.

9. A rectangle and a triangle have equal bases. The altitude of the rectangle is 25 feet and of the triangle is 20 feet. The combined area of the triangle and the rectangle is 280 square feet. Find the length of the base.

10. Find two consecutive positive odd integers whose squares differ by 32.

11. The length of a rectangular lot is three times its width. If the length is decreased by 20 feet and the width is increased by 10 feet, the area is increased by 200 square feet. Find the original dimensions.

12. A triangle and a rectangle have the same base. The altitude of the rectangle is 4 feet longer, and the altitude of the triangle is 5 feet shorter than the base. The area of the rectangle is 90 square feet greater than twice the area of the triangle. Find the length of the base.

13. A sum of money amounting to $13.55 consists of nickels, dimes, and quarters. There are three times as many dimes as nickels and three less quarters than dimes. How many of each coin are there?

14. If $(59 - 3x)$ is divided by the integer x, the quotient is 5 and the remainder is 3. Find the value of x.

HINT. Recall: **dividend = (quotient) · (divisor) + remainder.**

15. A peddler sold 7 bushels more than $\frac{5}{8}$ of his load of apples and then had 9 bushels less than $\frac{2}{5}$ of the load remaining. Find his original load.

16. In 1 hour, Jones can plow $\frac{1}{8}$ of a field and Roberts $\frac{1}{12}$ of it. If they work together, how long will it take them to plow the field?

17. A room can be painted in 21 hours by Smith and in 14 hours by Johnson. How long will it take them to paint the room working together?

18. How long will it take two mechanical ditchdiggers to excavate a ditch which the first machine, alone, could complete in 8 days and the second, alone, in 11 days?

19. How long will it take workers A and B, together, to complete a job which could be done by A alone in 7 days, and by B alone in 9 days?

20. How long will it take to fill a reservoir with intake pipes A, B, and C open, if the reservoir could be filled through A alone in 6 days, B alone in 8 days, and C alone in 5 days?

21. If 1000 articles of a given type can be turned out by a first machine in 9 hours, by a second in 6 hours, and by a third in 12 hours, how long will it take to turn out the articles if all machines work together?

22. An integer between 10 and 100 ends in 5. By writing an equation, find the integer if it is 5 times the sum of its digits.

61. Percentage

The words *per cent* are abbreviated by the symbol % and mean *hundredths*. That is, if r is the value of $h\%$, then

$$h\% = \frac{h}{100} = r. \tag{1}$$

ILLUSTRATION 1. $6\% = \dfrac{6}{100} = .06.$ $4\frac{3}{4}\% = \dfrac{4.75}{100} = .0475.$

From equation 1, we obtain $h = 100r$; hence, to change a number to per cent form, we *multiply* r *by* 100 *and add the* % *symbol.*

ILLUSTRATION 2. If $r = .0175$, then $100r = 1.75$ and $.0175 = 1.75\%$.

ILLUSTRATION 3. $$\frac{18}{800} = \frac{9}{400} = .0225 = 2.25\%.$$

The description of a ratio M/N in per cent form is the background for the following terminology.

If M is described by the relation $M = Nr$, where r is the ratio of M to N, we sometimes say that M is expressed as a *percentage* of N, with r as the **rate** and N as the **base** for the percentage:

$$\boldsymbol{M = Nr}, \qquad \text{or} \qquad \textbf{percentage} = \textbf{(base)} \cdot \textbf{(rate)}; \tag{2}$$

$$\boldsymbol{r = \frac{M}{N}}, \qquad \text{or} \qquad \textbf{rate} = \frac{\textbf{percentage}}{\textbf{base}}. \tag{3}$$

ILLUSTRATION 4. To express 375 as a percentage of 500, we compute the rate $r = \frac{375}{500} = .75$. Hence, $375 = .75(500)$, or 375 is 75% of 500.

EXAMPLE 1. Find the number of residents of a city where 13% of the population, or 962 people, had influenza.

SOLUTION. Let P be the number of residents:

$$.13P = 962; \qquad P = \frac{962}{.13} = \frac{96{,}200}{13} = 7400.$$

Note 1. In the formation of a mixture of different ingredients, we shall assume that there is no change in volume. Actually, a slight gain or loss of volume might occur, for instance, as a result of chemical action.

In the typical mixture problem, where one special ingredient is involved, the equation for solving the problem frequently can be obtained by writing, in algebraic form, the statement that

$$\left\{\begin{matrix} \textit{the sum of the amounts of the} \\ \textit{ingredient in the parts} \end{matrix}\right\} = \left\{\begin{matrix} \textit{amount of the ingredient} \\ \textit{in the final mixture.} \end{matrix}\right\} \tag{4}$$

If the *price* of a mixture is the fundamental feature, the equation may be obtainable by using equation 4 with the *costs* of the parts thought of in place of the *ingredients*.

EXAMPLE 2. How many gallons of a mixture containing 80% alcohol should be added to 5 gallons of a 20% solution to give a 30% solution?

SOLUTION. 1. Let x be the number of gal. added. In x gal., 80% pure alcohol, there are $.80x$ gal. of alcohol.

2. In 5 gal., 20% pure, there are .20(5) gal. of alcohol.

3. In $(5 + x)$ gal., 30% pure, there are $.30(x + 5)$ gal. of alcohol.

4. The alcohol in the final mixture of $(5 + x)$ gal. is the sum of the alcohol in the x gal. and in the 5 gal. Or,

$$.30(x + 5) = .80x + .20(5); \qquad x = 1.$$

EXAMPLE 3. What percentage of a 20% solution of hydrochloric acid should be drawn off and replaced by water to give a 15% solution?

SOLUTION. 1. Think of the solution as consisting of 100 units of volume; then the solution contains 20 units of acid.

2. Let $x\%$ be the rate for the percentage which should be drawn off. Then, from the 100 units we should draw off $x\%$ of 100, or x units.

3. In x units there are $.20x$ units of acid. There remain $(20 - .20x)$ units in the final solution of 100 units, after water is added. Therefore,

$$.15 = \frac{20 - .20x}{100}; \qquad 15 = 20 - .20x; \qquad x = 25.$$

Or, we should draw off 25% of the original solution.

EXERCISE 24

Change to decimal form.

1. 5%. **2.** $4\frac{1}{4}\%$. **3.** $3\frac{3}{4}\%$. **4.** 45%. **5.** 126.3%. **6.** $\frac{5}{12}\%$.

Change to per cent form.

7. .07. **8.** .0925. **9.** .025. **10.** .0575. **11.** 1.35.

Compute each quantity.

12. 6% of \$300. **13.** $3\frac{1}{4}\%$ of 256. **14.** 110% of 1250.

Express the first number as a percentage of the second.

15. 75, of 200. **16.** 400, of 640. **17.** 350, of 200.

18. The average price of copper per pound in the United States was approximately \$.138 in 1926, \$.081 in 1931, and \$.215 in late 1947. Find the per cent of change in the price from 1926 to 1931; from 1931 to 1947.

Solve each problem by using an equation in just one unknown.

19. If 385 is 85% of x, find x. **20.** If 268 is 24% less than y, find y.

21. After selling 85% of a stock of dresses, a merchant finds that he has 84 dresses left. What was his original stock?

22. A merchant buys 100 dozen shirts at $13.20 per dozen. He sells 90 dozen at a markup of 30% over the purchase price. At what price per shirt could he afford to sell the remaining 10 dozen to clear his stock if he desires his total receipts from the shirts to be 25% greater than the cost?

23. $3000 of Smith's income is not taxed by the state where he lives. All of his income over $3000 is taxed 2% and all over $8000 is taxed 3% in addition. If he pays a total tax of $800, what is his income?

24. Under the taxes of Problem 23, with an additional surtax of 5% on all income over $20,000, Johnson's tax is $1400. Find his income.

25. A merchant has some coffee worth 70¢ per pound and some worth 50¢. How many pounds of each are used in forming 100 pounds of a mixture worth 65¢ per pound?

26. How many gallons of a mixture of water and alcohol containing 60% alcohol should be added to 9 gallons of a 20% solution to give a 30% solution?

27. How many gallons of a solution of glycerine and water containing 55% glycerine should be added to 15 gallons of a 20% solution to give a 40% solution?

28. How many ounces of pure silver must be added to 150 ounces, 45% pure, to give a mixture containing 60% silver?

29. A feed merchant wishes to form 200 bushels of a mixture of wheat at $1.25 per bushel and wheat at $.80 per bushel, so that the mixture will be worth $1.00 per bushel. How much of each kind should he use?

30. How many pounds of cream containing 35% butterfat should be added to 800 pounds of milk containing 3% butterfat to give milk containing 3.5% butterfat?

31. An automobile radiator holds 8 gallons of a solution containing 40% glycerine. How much of the solution should be drawn off and replaced by water to give a solution with 25% glycerine?

32. What percentage of a 30% solution of sulphuric acid should be drawn off and replaced by water to give a 20% solution?

33. What percentage of a 40% solution of alcohol and water should be replaced by pure alcohol, to give a 75% solution?

34. What percentage of a mixture of sand, gravel, and cement, containing 30% cement, should be replaced by pure cement in order to give a mixture containing 40% cement?

62. Lever problems

A lever consists of a rigid rod with one point of support called the *fulcrum.* A familiar instance of a lever is a teeterboard. If a weight w is attached to a lever at a certain point, the distance h of w from the fulcrum is called the *lever arm* of w, and the product hw is called the *moment* of w about the fulcrum. The following statement is demonstrated in physics.

LEVER PRINCIPLE. *If two or more weights are placed along a lever in such a way that the lever is in a position of equilibrium, then, if each weight is multiplied by its lever arm, the sum of these products for all weights on one side of the fulcrum equals the sum of the products for all weights on the other side. In other words, the sum of the moments of the weights about the fulcrum is the same on both sides.*

Note 1. In all lever problems in this book, it will be assumed that the weight of the lever is negligible for the purpose in view.

ILLUSTRATION 1. In Figure 2, the sum of the moments for the weights at the left is $(5 \cdot 80 + 4 \cdot 250)$ or 1400, and for those at the right is

$$(4 \cdot 100 + 5 \cdot 200), \text{ or } 1400.$$

Hence, this lever is balanced.

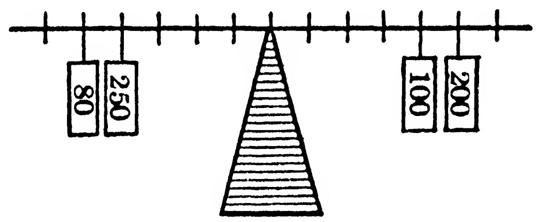

Fig. 2

EXAMPLE 1. Two girls, weighing 75 pounds and 90 pounds, respectively, sit at the ends of a teeterboard 15 feet long. Where should the fulcrum be placed to balance the board?

SOLUTION. 1. Let x feet be the distance from the fulcrum to the lighter girl. Then, the lever arm for the other girl is $(15 - x)$ feet.

2. Hence, from Figure 3,

$$75x = (15 - x)90;$$
$$x = 8\tfrac{2}{11} \text{ feet.}$$

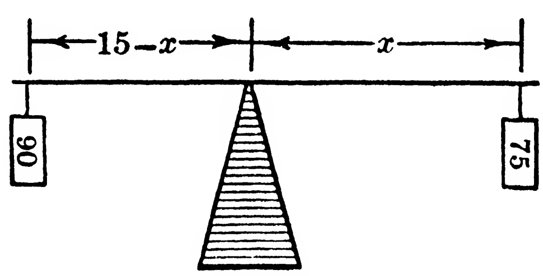

Fig. 3

EXERCISE 25

1. A weight of 300 pounds is placed on a lever 20 feet from the fulcrum. Where should a weight of 275 pounds be placed to balance the lever?

2. A weight of 60 pounds is placed on a lever 8 feet from the fulcrum. How heavy a weight should be placed 12 feet from the fulcrum on the other side to give equilibrium?

3. A teeterboard is balanced when one girl weighing 80 pounds sits 4 feet from the fulcrum, another girl weighing 100 pounds sits 7 feet from the fulcrum on the other side, and a third girl sits 6 feet from the fulcrum. How much does the third girl weigh?

4. Jones and Smith together weigh 340 pounds. Find their weights if they balance a lever when Jones sits 5 feet from its fulcrum on one side and Smith sits 6 feet from the fulcrum on the other side.

5. A 40-pound weight is placed 6 feet from the fulcrum on a lever, and a 60-pound weight 8 feet from the fulcrum on the other side. Where should a 30-pound weight be placed to give equilibrium?

6. How heavy a weight can a man lift with a lever 9 feet long if the fulcrum is 2 feet from the end under the weight and if the man exerts a force of 140 pounds on the other end?

7. How many pounds of force must a man exert on one end of an 8-foot lever to lift a 300-pound rock on the other end if the fulcrum is $1\frac{1}{2}$ feet from the rock?

63. Uniform motion

When we say that a body is moving in a path at *constant speed*, we mean that the body passes over *equal distances in any two equal intervals of time*. Such motion is referred to as *uniform motion* in the path. The *velocity* * or *speed* or *rate* of the body in its path is defined as the distance traveled in one unit of time. If v is the velocity, and d is the distance traveled in t units of time,

$$d = vt. \tag{1}$$

Since $v = \dfrac{d}{t}$, the velocity is referred to as *the rate of change of the distance with respect to the time*. In stating a velocity, the units for the measurement of time and of distance must always be mentioned.

ILLUSTRATION 1. If an airplane flies 1250 miles in 5 hours at uniform speed, the speed is

$$v = \frac{d}{t} = \frac{1250}{5}, \quad \text{or} \quad 250 \text{ miles per hour.}$$

The speed of the airplane per minute is $\frac{250}{60}$ or $4\frac{1}{6}$ miles.

* In physics, *velocity* is defined as a vector quantity, possessing both *magnitude* and *direction*. In this text, wherever the word *velocity* is used, it will refer to the *magnitude* (positive) of the velocity vector.

Note 1. All motion considered in this book will be *uniform* motion. If the velocity of a moving body is variable, a discussion of the motion must bring in more advanced notions met in physics and calculus.

EXAMPLE 1. A messenger, traveling at a speed of 65 miles per hour, pursues a truck which has a start of 2 hours and overtakes the truck in 3 hours. Find the speed of the truck.

SOLUTION. Let x miles per hour be the truck's speed. Then,

$$3\cdot(65) = (3 + 2)x; \qquad 195 = 5x; \qquad x = 39 \text{ miles per hour.}$$

The equation $d = rt$ applies in the discussion of any variable quantity d which changes *uniformly* at a specified rate r with respect to change in the time t. Thus, we may refer to a rate of *increase* or a rate of *decrease* under various conditions.

EXAMPLE 2. A motorboat went 70 miles in 4 hours when traveling at full speed *upstream* on a river whose current flows at the rate of 6 miles per hour. How fast can the boat travel in still water?

SOLUTION. 1. Let x miles per hour be the speed of the boat in still water. In travel upstream, the rate of the current *reduces* the speed of the boat to $(x - 6)$ miles per hour.

2. From $d = vt$, with $t = 4$, $v = x - 6$, and $d = 70$,

$$70 = 4(x - 6). \tag{2}$$

On solving (2) we obtain $\qquad 4x = 94; \qquad x = 23.5.$

Hence, the boat can travel 23.5 miles per hour in still water.

64. Radius of action of an airplane

Suppose that an airplane flies with a *groundspeed* * of G_1 miles † per hour in a particular direction from a base B and then back along this path at a groundspeed of G_2 miles per hour, when the engines are working at full power, and while the wind maintains a constant direction and speed. G_1 and G_2 in general would be different because of the effects of the wind. Suppose that the gasoline tanks of the airplane permit it to operate at full power for only T hours after leaving B. We refer to T as the available *fuel hours*. Then, it is of interest to consider the maximum length of time, h hours, during which the airplane may fly out if it is to return to B by the end of

* Speed with respect to the *ground* as contrasted to the *airspeed*, or speed with respect to the *air*, which itself may be in motion because of a wind.

† The subscript 1 on G_1 is just a tag. We read "G_1" as "G sub 1" or "G, 1."

T hours. The distance R which the airplane flies out from B in h hours is called the *radius of action* of the airplane in the specified direction, with the given wind. It can be proved * that

$$h = \frac{TG_2}{G_1 + G_2} \quad \text{and} \quad R = \frac{TG_1G_2}{G_1 + G_2}. \tag{1}$$

Example 1. How many fuel hours must be available in order to have 900 miles as the radius of action in a direction where the groundspeeds out and back are, respectively, 300 and 200 miles per hour?

Solution. Substitute $R = 900$, $G_1 = 300$, and $G_2 = 200$ in the second equation in (1):

$$900 = \frac{T(300)(200)}{300 + 200}, \quad \text{or} \quad 900 = 120T.$$

Hence,
$$T = \frac{900}{120} = 7\tfrac{1}{2} \text{ hr.}$$

EXERCISE 26

1. At what rate does an automobile travel if it goes 450 miles in 9 hours?

2. Jones and Smith travel toward each other from points 500 miles apart, Jones at the rate of 60 miles per hour and Smith at 50 miles per hour. When will they meet if they start at the same instant?

3. Two men start at 7 A.M. from the same place, in opposite directions, at speeds of 36 miles and 48 miles per hour, respectively. When will they be 600 miles apart?

4. At 6 A.M., a motorcycle messenger starts from a city at a speed of 45 miles per hour to meet a regiment which is 120 miles away and is approaching at a speed of 5 miles per hour. When will the messenger meet the regiment?

5. One man can run 400 meters in 54 seconds and a second man can run the distance in 60 seconds. How long will it take the faster man to gain a lead of 12 meters on the slower man if they start together in a 400-meter race?

6. An airplane leaves the deck of a battleship and travels south at the rate of 230 miles per hour. The battleship travels south at the rate of 20 miles per hour. If the wireless set on the airplane has a range of 800 miles, when will the airplane pass out of wireless communication with the ship?

* See page 51, in William L. Hart's *College Algebra, 3d Edition*, D. C. Heath and Company.

7. How many seconds will it take for a man to travel y miles if he travels x miles in t seconds?

8. In an 800-meter race between two men, the winner's time is 2 minutes, and his lead is 40 meters. How many seconds would it take the loser to run 800 meters?

9. An airplane flew 850 miles in $2\frac{1}{2}$ hours against a head wind blowing 30 miles per hour. How fast could the plane fly in still air?

10. Two men start together in a race around a 300-yard oval track, one man at a speed of 9 yards per second and the other man at $7\frac{1}{2}$ yards per second. When will the faster man be exactly one lap ahead?

11. When will Johnson be twice as wealthy as Smith if each has $4000 now and if their estates are increasing at the annual rates of $400 for Smith and $1200 for Johnson?

12. Jones can run around a 400-meter track in 65 seconds. How long does Smith take to run the 400 meters if he meets Jones in 35 seconds after they start together in a race around the track in opposite directions?

In each problem, (a) find the radius of action for a flight by an airplane in a direction where the groundspeeds have the indicated values; (b) find the number of hours flown on the maximum outward trip.

13. Sixteen fuel hours available; groundspeed out is 240 miles and back is 210 miles per hour.

14. Twelve fuel hours available; groundspeed out is 190 miles and back is 225 miles per hour.

15. Fourteen fuel hours available; groundspeed out is 175 miles and back is 200 miles per hour.

If an airplane is to have the specified radius of action in a direction corresponding to the given groundspeeds, find the number of fuel hours which must be available.

16. Radius of action is 1350 miles; groundspeed out is 200 miles and back is 225 miles per hour.

17. Radius of action is 750 miles; groundspeed out is 195 miles and back is 175 miles per hour.

18. How many fuel hours must be available to permit an airplane flight out from a field for $5\frac{1}{4}$ hours in a direction such that the groundspeed out is 200 miles and back is 185 miles per hour?

19. At how many minutes after 2 P.M. will the minute hand of a clock overtake the hour hand?

20. After 10 P.M., when will the hands of a clock first form a straight line?

65. Interest

Interest is income received from invested capital. The capital originally invested is called the **principal.** At any time after the investment of the principal, the sum of the principal and the interest due is called the **amount.**

The **rate of interest** is the ratio of the interest earned in one year to the principal. If r is the rate and P is the principal, then

$$r = \frac{\text{interest per year}}{P}; \tag{1}$$

$$\text{interest for one year} = Pr. \tag{2}$$

Thus, the interest is a *percentage of the principal,* with r as the rate.

ILLUSTRATION 1. If \$1000 earns \$36.60 interest in one year, then, from equation 1,

$$r = \frac{36.60}{1000} = .0366, \quad \text{or} \quad r = 3.66\%.$$

66. Simple interest

If interest is computed on the *original* investment during the *whole life* of a transaction, the interest earned is called *simple interest.*

Suppose that P is invested at simple interest for t years at the rate r. Let I be the interest and F be the final amount at the end of the t years. Then, the interest for one year is Pr and, by definition, the simple interest for t years is $t(Pr)$ or Prt; that is,

$$I = Prt. \tag{1}$$

Since *amount equals principal plus interest,*

$$F = P + I. \tag{2}$$

From (1),

$$P + I = P + Prt = P(1 + rt).$$

Hence, from (2),

$$F = P(1 + rt). \tag{3}$$

In equations 1 and 3, t represents the time expressed in years. If the time is described in *months,* we express it in years assuming a year to contain 12 equal months. If the time is given in *days,* there are two varieties of interest used, **ordinary** and **exact simple interest.** In computing ordinary interest we assume a year to contain 360 days, and, for exact interest, we assume a year to contain 365 days.

To find the amount F when P, r, and t are given, first find the interest from $I = Prt$ and then compute $P + I$ to find F.

Note 1. Unless otherwise specified, the word "*interest*" in this book will refer to *simple* interest.

ILLUSTRATION 1. If \$5000 is invested for 59 days at 5%,

the ordinary interest due is $\quad 5000(.05)\frac{59}{360} = \$40.97;$

the final amount due is $\quad 5000 + 40.97 = \$5040.97.$

EXAMPLE 1. If \$1000 accumulates to \$1250 when invested at simple interest for 3 years, find the interest rate.

SOLUTION. 1. We have $P = \$1000$; $F = \$1250$; $I = 1250 - 1000 = \$250$.

2. From $I = Prt$ with $t = 3$,

$$250 = 1000(r)(3); \qquad 250 = 3000r; \qquad r = \frac{250}{3000} = .08\tfrac{1}{3} = 8\tfrac{1}{3}\%.$$

In $F = P(1 + rt)$, the principal P is frequently called the *present value* of the amount F because, if P is invested today at the rate r, the accumulated amount at the end of t years will be F.

EXAMPLE 2. Find the present value of \$1100 which is due at the end of $2\frac{1}{2}$ years, if money can be invested at 4%.

SOLUTION. 1. We have $F = \$1100$, $r = .04$, and $t = 2\frac{1}{2}$.

2. Hence, from (3),

$$1100 = P[1 + \tfrac{5}{2}(.04)]; \quad 1100 = P(1 + .10);$$

$$1.1P = 1100; \quad P = \frac{1100}{1.1} = \$1000.$$

CHECK. $\quad I = 1000(.04)(\frac{5}{2}) = \$100; \quad F = 1000 + 100 = \$1100.$

EXERCISE 27

Find the ordinary interest and the final amount.

1. On \$5000 at 6% for 216 days.
2. On \$8000 at $4\frac{1}{2}\%$ for 93 days.

Find the exact interest and the final amount.

3. On \$3000 at 4% for 146 days.
4. On \$2500 at $5\frac{1}{2}\%$ for 27 days.
5. Find the amount due at the end of 8 months if \$150 is invested at 9%.

With the given data, solve $F = P(1 + rt)$ for P, to the nearest cent.

6. $F = \$1000$; $r = .03$; $t = \frac{5}{7}$.
7. $F = \$3000$; $r = .05$; $t = \frac{7}{12}$.

8. At what rate will \$750 be the interest for 5 years on \$6000?

9. Find the invested principal if it earns \$375 interest in 3 months when the interest rate is $3\frac{1}{2}\%$.

10. Find the principal if it earns \$150 interest in $\frac{1}{2}$ year at 8%.

11. (*a*) Find the principal which will amount to \$1300 by the end of 6 years when invested at 5%. (*b*) Verify the result by computing interest on it for 6 years.

12. Find the present value of \$1888 which is due at the end of 4 years, if the interest rate is $4\frac{1}{2}\%$.

13. Jones agreed to pay Smith \$6000 at the end of 5 years. What should Jones pay immediately to cancel his debt if Smith agrees that he can invest money at 4%?

14. Roberts buys a bill of goods from a merchant who asks \$2000 at the end of 2 months. If Roberts wishes to pay immediately, what should the seller be willing to accept if he is able to invest his money at 8%?

15. A debtor owes \$1100 due at the end of 2 years and he requests the privilege of paying an equivalent smaller sum immediately. At what simple interest rate would the creditor prefer to compute the present payment, at 5% or at 6%, and how much would he gain by the best choice?

16. How long will it take a given principal to double itself if invested at 5% simple interest?

17. A man invests \$7000, one part of it at 5% and the balance at 4%. If the total annual interest is \$320, how much is invested at each rate?

CHAPTER 5

SPECIAL PRODUCTS AND FACTORING

67. Square root

If $R^2 = A$, we call R a **square root** of A.

ILLUSTRATION 1. 4 is a square root of 16 because $4^2 = 16$.

Every positive number A has *two* square roots, one *positive* and one *negative*, with equal absolute values. The positive square root is denoted by $+\sqrt{A}$, or simple $\sqrt{A}$, and the negative square root by $-\sqrt{A}$. We call $\sqrt{A}$ a **radical** and A its **radicand.** Unless otherwise stated, *the* square root of A will mean its *positive* square root. By the definition of a square root,

$$(\sqrt{A})^2 = A. \tag{1}$$

If x is positive or zero,

$$\sqrt{x^2} = x. \tag{2}$$

ILLUSTRATION 2. $\sqrt{9} = 3$ because $3^2 = 9$. The two square roots of 9 are $\sqrt{9}$ and $-\sqrt{9}$, or $\pm\sqrt{9}$, or ± 3. We read "$\pm$" as "*plus or minus.*"

ILLUSTRATION 3. $\sqrt{\frac{1}{4}} = \frac{1}{2}$ because $(\frac{1}{2})^2 = \frac{1}{4}$.

68. Perfect squares

In the square of an integral rational term, each exponent will be an *even* integer because, in squaring, each original exponent is multiplied by 2.

ILLUSTRATION 1. $(3x^2y^3)^2 = 3^2(x^2)^2(y^3)^2 = 9x^4y^6.$

An *integer* is said to be a **perfect square** if it is the square of an *integer*. The student should learn the most common perfect square integers, with the aid of Table I, page 283.

ILLUSTRATION 2. The perfect square integers are 1, 4, 9, 16, 25, 36, etc. Their square roots are, respectively, 1, 2, 3, 4, 5, 6, etc. Thus, $\sqrt{25} = 5$ because $5^2 = 25$.

An integral rational term is said to be a *perfect square* if it is the square of some other term of the same variety. Hence, in a perfect square, each exponent is an *even* integer.

ILLUSTRATION 3. $25a^2b^4$ is a perfect square because $25a^2b^4 = (5ab^2)^2$.

I. *To find the square root of a perfect square monomial:*

1. *Rewrite the literal part with each exponent divided by* 2.

2. *Multiply by the square root of the numerical coefficient of the given term.*

ILLUSTRATION 4. $\sqrt{16x^4y^8} = \sqrt{16}\sqrt{x^4y^8} = 4x^2y^4,$ because $(4x^2y^4)^2 = 4^2(x^2)^2(y^4)^2 = 16x^4y^8.$

II. *To find the square root of a fraction, find the square root of the numerator and of the denominator and divide:*

$$\sqrt{\frac{N}{D}} = \frac{\sqrt{N}}{\sqrt{D}}. \tag{1}$$

ILLUSTRATION 5. $\sqrt{\frac{4}{25}} = \frac{\sqrt{4}}{\sqrt{25}} = \frac{2}{5}.$ $\sqrt{\frac{a^2}{b^2}} = \frac{\sqrt{a^2}}{\sqrt{b^2}} = \frac{a}{b}.$

ILLUSTRATION 6. $\sqrt{\frac{100a^6}{9x^4y^8}} = \frac{\sqrt{100a^6}}{\sqrt{9x^4y^8}} = \frac{10a^3}{3x^2y^4}.$

Note 1. For the present, we shall consider $\sqrt{A}$ only where A is a perfect square monomial, or where A is a fraction whose numerator and denominator are perfect square monomials. All literal numbers in A will be supposed positive or zero.

EXERCISE 28

Find the two square roots of the number and check by squaring the results.

1. 25. **2.** 49. **3.** 121. **4.** 64. **5.** $\frac{1}{9}$. **6.** $\frac{1}{36}$.

Find each square root and check by squaring the result. Inspect Table I *if necessary.*

7. $\sqrt{9}$. **8.** $\sqrt{100}$. **9.** $\sqrt{81}$. **10.** $\sqrt{144}$.

11. $\sqrt{196}$.
12. $\sqrt{\frac{9}{4}}$.
13. $\sqrt{\frac{16}{25}}$.
14. $\sqrt{\frac{25}{9}}$.
15. $\sqrt{\frac{1}{81}}$.
16. $\sqrt{\frac{49}{64}}$.
17. $\sqrt{\frac{4}{81}}$.
18. $\sqrt{\frac{49}{100}}$.
19. $\sqrt{x^4}$.
20. $\sqrt{y^6}$.
21. $\sqrt{a^2}$.
22. $\sqrt{w^8}$.
23. $\sqrt{a^{12}}$.
24. $\sqrt{z^{10}}$.
25. $\sqrt{4a^4}$.
26. $\sqrt{9y^4}$.
27. $\sqrt{4z^8}$.
28. $\sqrt{16h^2}$.
29. $\sqrt{49z^6}$.
30. $\sqrt{9x^2y^4}$.
31. $\sqrt{64x^4w^6}$.
32. $\sqrt{16a^2b^8}$.
33. $\sqrt{49w^4x^4}$.
34. $\sqrt{121z^2}$.
35. $\sqrt{\frac{9}{a^2}}$.
36. $\sqrt{\frac{4}{x^2}}$.
37. $\sqrt{\frac{y^2}{16}}$.
38. $\sqrt{\frac{z^2}{25}}$.
39. $\sqrt{\frac{49}{w^4}}$.
40. $\sqrt{\frac{121}{y^6}}$.
41. $\sqrt{\frac{a^2}{y^4}}$.
42. $\sqrt{\frac{b^4}{w^8}}$.
43. $\sqrt{\frac{4x^2}{h^8}}$.
44. $\sqrt{\frac{a^4}{9x^2}}$.
45. $\sqrt{\frac{81a^2}{y^2z^2}}$.
46. $\sqrt{\frac{b^2x^4}{49z^4}}$.
47. $\sqrt{\frac{9a^2b^4}{c^6w^{10}}}$.
48. $\sqrt{\frac{9x^6}{4y^4z^2}}$.
49. $\sqrt{\frac{121a^2}{9b^4z^6}}$.
50. $\sqrt{\frac{100x^4}{9y^2z^6}}$.

Square each quantity.

51. $(\sqrt{37})^2$.
52. $(\sqrt{142a})^2$.
53. $(\sqrt{yz^3})^2$.
54. $(\sqrt{659})^2$.

55. A negative number can have neither a positive nor a negative square root. Why is this true?

69. Products of binomials

When desired, the product of two binomials may be found by longhand methods.

ILLUSTRATION 1. $(3x - 5y)(2x - 7y) = 3x(2x - 7y) - 5y(2x - 7y)$
$$= 6x^2 - 21xy - 10xy + 35y^2 = 6x^2 - 31xy + 35y^2.$$

ILLUSTRATION 2. $(x + y)(x - y) = x(x - y) + y(x - y)$
$$= x^2 - xy + xy - y^2 = x^2 - y^2.$$

ILLUSTRATION 3. $(a + b)^2 = (a + b)(a + b)$
$$= a(a + b) + b(a + b) = a^2 + 2ab + b^2.$$

70. Special products

The student should be able to dispense with the longhand methods of the preceding section and should form the product of two binomials mentally. Products of the following types occur frequently. The student should verify each right-hand member.

I.	$a(x + y) = ax + ay.$
II.	$(x + y)(x - y) = x^2 - y^2.$
III.	$(a + b)^2 = a^2 + 2ab + b^2.$
IV.	$(a - b)^2 = a^2 - 2ab + b^2.$
V.	$(x + a)(x + b) = x^2 + (ax + bx) + ab.$
VI.	$(ax + b)(cx + d) = acx^2 + (adx + bcx) + bd.$

It proves convenient to memorize Types II, III, and IV as formulas and also in words.

ILLUSTRATION 1. Type II states that *the product of the sum and the difference of two numbers is the difference of their squares.*

ILLUSTRATION 2. Type III states that *the square of the sum of two numbers equals the square of the first, plus twice the product of the numbers, plus the square of the second number.*

ILLUSTRATION 3. $(c - 2d)(c + 2d) = c^2 - (2d)^2 = c^2 - 4d^2.$ (Type II)

ILLUSTRATION 4. From Type III with $a = 3x$ and $b = 2y$,

$$(3x + 2y)^2 = (3x)^2 + 2(3x)(2y) + (2y)^2 = 9x^2 + 12xy + 4y^2.$$

The right members of Types V and VI should not be committed to memory. However, the *nature* of the right members should be memorized, with $(ax + bx)$ in Type V and $(adx + bcx)$ in Type VI remembered as *the sum of the cross products.*

ILLUSTRATION 5. To obtain $(2x - 5)(3x + 7)$:

$$\left.\begin{matrix} 2x & -5 \\ 3x & +7 \end{matrix}\right\} \textit{Product} = 6x^2 - x - 35.$$

Sum of the cross products: $14x - 15x = -x.$

The diagram and auxiliary computation of the sum of the cross products should be omitted and replaced by mental computation as in the next illustration.

ILLUSTRATION 6. $(2x - 7h)(3x + 2h) = 6x^2 - 17hx - 14h^2,$

because the sum of cross products is $-21hx + 4hx$, or $-17hx$.

ILLUSTRATION 7. $(x^2 - 3y^3)^2 = (x^2)^2 - 2(x^2)(3y^3) + (3y^3)^2$ (Type IV)

$$= x^4 - 6x^2y^3 + 9y^6.$$

ILLUSTRATION 8. $(x^2 - 2y)(x^2 + 2y)(x^4 + 4y^2)$ (Type II)

$$= [(x^2 - 2y)(x^2 + 2y)](x^4 + 4y^2) = (x^4 - 4y^2)(x^4 + 4y^2)$$
$$= (x^4)^2 - (4y^2)^2 = x^8 - 16y^4.$$

ILLUSTRATION 9. $(-3x - 4)(-3x + 4) = -(4 + 3x)(4 - 3x)$

$$= -(16 - 9x^2) = -16 + 9x^2.$$

EXERCISE 29

Expand and collect terms, performing as much of the work as possible mentally.

1. $5(3a - 4v)$. **2.** $3c(2 - 6c)$. **3.** $ab(4x - ax)$.

4. $-5x(2y - 3x)$. **5.** $(c - d)(c + d)$. **6.** $(h - 2k)(h + 2k)$.

7. $(a + y)^2$. **8.** $(c - 3x)(c + 3x)$. **9.** $(4 - y)(4 + y)$.

10. $(5 - 2y)(5 + 2y)$. **11.** $(3 + 2r)(3 - 2r)$.

12. $(3x - 4z)(3x + 4z)$. **13.** $(a^2 - 3b)(a^2 + 3b)$.

14. $(ab - 2)(ab + 2)$. **15.** $(a - 2)(a - 4)$.

16. $(c + 3)^2$. **17.** $(x + 5)^2$. **18.** $(y - 4)^2$.

19. $(2a - 5)^2$. **20.** $(3x - 2)^2$. **21.** $(2z - w)^2$.

22. $(3x - 4y)^2$. **23.** $(2a + b)^2$. **24.** $(x - 3y)^2$.

25. $(3 + x)(2 + x)$. **26.** $(2x + 5y)(-2x - 5y)$.

27. $(x - 5)(x + 9)$. **28.** $(x + 13)(x - 4)$.

29. $(a + 2b)(a + 3b)$. **30.** $(w - 2z)(w + 5z)$.

31. $(2x + 3)(3x + 4)$. **32.** $(3x - 5)(2x - 3)$.

33. $(4y - 3x)(2y - x)$. **34.** $(2y + w)(y + 5w)$.

35. $(2y - 3)(3y + 5)$. **36.** $(y - 3)(2y + 7)$.

37. $(3w + 5)(7w - 2)$. **38.** $(3 - 4x)(2 + 5x)$.

39. $(4x - 3y)(2x + 3y)$. **40.** $(3u - 5w)(2u + 3w)$.

41. $(6 - 5x)(-2 + x)$. **42.** $(-3 - 2x)(2 - x)$.

43. $(3x - 4)(-x + 5)$. **44.** $(-y - 3)(2y + 4)$.

45. $(x^2 + 2)^2$. **46.** $(4 + 3b^2)^2$. **47.** $(2xy - 3y^2)^2$.

48. $(4ax^2 - y)^2$. **49.** $(3 + 4bx)^2$. **50.** $(x - 2wx^2)^2$.

51. $(x - \frac{1}{3})^2$. 52. $(y + \frac{1}{4})^2$. 53. $(\frac{1}{3} - 2z)^2$.

54. $(wx^2 - 2a)(wx^2 + 2a)$. 55. $(c^3d - 3x^2)(c^3d + 3x^2)$.

56. $(x + .1)(x + .5)$. 57. $(x - .2)(x + .5)$.

58. $(.3 + x)(.2 - x)$. 59. $(3 - .2x)(2 + .5x)$.

60. $(\frac{2}{3}a - b)(\frac{2}{3}a + b)$. 61. $(\frac{1}{2}a - \frac{1}{3}b)(\frac{1}{2}a + \frac{1}{3}b)$.

62. $(\frac{3}{4}x + \frac{2}{5}y)(\frac{3}{4}x - \frac{2}{5}y)$. 63. $(.4x - .3)(.2x - .5)$.

64. $(x - y)(x + y)(x^2 + y^2)$. 65. $(2 - x^2)(2 + x^2)(4 - x^4)$.

66. $(w - 3)(w + 3)(w^2 + 9)$. 67. $-7x(2ax - 3x^2 - x^3)$.

68. $-3yz(2y^2 - 3yz + z^2)$. 69. $(-3 - 4x)(-2 + 5x)$.

70. $(-x - y)^2$. 71. $(-2x - 3y)^2$. 72. $(-3 - 5z)^2$.

73. $[2(x - y)]^2$. 74. $[3(a + b)]^2$. 75. $[5(2c - 3d)]^2$.

76. $(3x^2 - 5)(x^2 + 2)$. 77. $(4x^2 - 3)(3x^2 + 2)$.

78. $(2x^2 - 3y^2)(x^2 + 4y^2)$. 79. $(2a^2 + 5b^2)(a^2 - 3b^2)$.

80. $(x^3 + 3)(3x^3 - 4)$. 81. $(3a^3 - 2b^3)(7a^3 + 5b^3)$.

82. $(2u^4 - 3v^2)(3u^4 + 2v^2)$. 83. $(4x^3 - 3y^4)(3x^3 + 2y^4)$.

84. $(2x^2 - 5y^2)(2x^2 + 5y^2)$. 85. $(3u^2 - 7v^2)(3u^2 + 2v^2)$.

86. $(2abc^2 - 5dw^3)(3abc^2 + 7dw^3)$.

71. Grouping in multiplication

The method of the following illustrations is particularly useful in applications of Types II, III, and IV of Section 70.

ILLUSTRATION 1. $(c + 2d - 11a)(c + 2d + 11a)$

$$= [(c + 2d) - 11a][(c + 2d) + 11a] \qquad \text{(Type II)}$$
$$= (c + 2d)^2 - (11a)^2 = c^2 + 4cd + 4d^2 - 121a^2.$$

ILLUSTRATION 2. $(2x + y - 3z)^2 = [(2x + y) - 3z]^2$ (Type IV)

$$= (2x + y)^2 - 2(3z)(2x + y) + (3z)^2$$
$$= 4x^2 + 4xy + y^2 - 12xz - 6yz + 9z^2.$$

ILLUSTRATION 3. $(a + b + c + 2) = [(a + b) + (c + 2)]^2$ (Type III)

$$= (a + b)^2 + 2(a + b)(c + 2) + (c + 2)^2$$
$$= a^2 + 2ab + b^2 + 2ac + 4a + 2bc + 4b + c^2 + 4c + 4.$$

ILLUSTRATION 4. $(2x - 3 + 2y)(2x + 3 - 2y)$

$$= [2x - (3 - 2y)][2x + (3 - 2y)]$$
$$= (2x)^2 - (3 - 2y)^2 = 4x^2 - 9 + 12y - 4y^2.$$

EXERCISE 30

Expand and collect terms by use of preliminary grouping.

1. $[(x + y) + 2]^2$. **2.** $[(a - b) + 5]^2$. **3.** $[3 - (2x - y)]^2$.

4. $(2 + a + w)^2$. **5.** $(3x + y + 5)^2$. **6.** $(x - 2y - 3)^2$.

7. $(4a - b - c)^2$. **8.** $(-2 + a + b)^2$. **9.** $[2x - 3(a - b^2)]^2$.

10. $(2x - 3x^2 + 3y)^2$. **11.** $[(x + y) - 3][(x + y) + 3]$.

12. $[(c + 2x) - 2][(c + 2x) + 2]$. **13.** $[4 - (2a + b)][4 + (2a + b)]$.

14. $(a + w + 4)(a + w - 4)$. **15.** $(a + b - x)(a + b + x)$.

16. $(3x + y - 2)(3x + y + 2)$. **17.** $(3a - y + 4)(3a - y - 4)$.

18. $(a - b^2 + z)(a + b^2 + z)$. **19.** $(x^2 - y + z)(x^2 + y - z)$.

20. $[(a + b) + (c - 3)]^2$. **21.** $[2x + y + a - 3]^2$.

22. $(a + c + b - 5)^2$. **23.** $(2x - z + y - 2)^2$.

24. $(a + b + c + d)(a + b - c - d)$.

25. $(2x + y - z + 3)(2x + y + z - 3)$.

26. $(a + 3y - 5z)(a + 3y - 4z)$.

27. $(c - 2d - a - x)(c - 2d + a + x)$.

28. Expand $(x + y + z)^2$ and state the result in words.

Use the formula of Problem 28 to expand each square.

29. $(2a + 3b + 4c)^2$. **30.** $(3a - 2b + 3c)^2$.

31. $(w - 5x + 3a)^2$. **32.** $(4z^3 - 3xy^2 - 5x^3)^2$.

72. Terminology about factoring

In our discussion of factoring, unless otherwise stated, the coefficients will be *integers* in any polynomial referred to. Such an expression will be called **prime** if it has no integral rational factors except *itself*, or its *negative*, or 1. No simple rule can be stated for determining whether or not an expression is prime.

ILLUSTRATION 1. We shall say that $(x - y)$ is prime although

$$x - y = (\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y}),$$

because these factors are not integral and rational. Other prime expressions are $(x + y)$, $(x^2 + y^2)$, $(x^2 + xy + y^2)$, and $(x^2 - xy + y^2)$.

To factor a polynomial will mean to express it as a product of positive integral powers of distinct *prime* factors.

ILLUSTRATION 2. To factor $4x^4 - 4b^2x^2$, we write

$$4x^4 - 4b^2x^2 = 4x^2(x^2 - b^2) = 4x^2(x - b)(x + b).$$

After an expression has been factored, the factors should always be verified by multiplying them to obtain the given expression.

73. Factoring by inspection

Each type formula of Section 70 becomes a formula for factoring when read *from right to left.*

I. $$ax + ay + az = a(x + y + z).$$

ILLUSTRATION 1. $$by^2 + 3y + 8y^2 = y(by + 3 + 8y).$$

ILLUSTRATION 2. If a factor $2x^2y^3$ is removed from the term $14x^3y^5$, the remaining factor can be verified by division:

$$14x^3y^5 \div 2x^2y^3 = 7xy^2.$$

Hence, $$14x^3y^5 = 2x^2y^3(7xy^2).$$

ILLUSTRATION 3. In the following factoring, we remove the common factor $2xy^2$ from each term:

$$14x^3y^5 + 6xy^2 - 8x^2y^3 = 2xy^2(7x^2y^3 + 3 - 4xy).$$

II. *The difference of two squares equals the product of the sum and the difference of their square roots:*

$$x^2 - y^2 = (x - y)(x + y).$$

ILLUSTRATION 4. $$x^2 - 9 = (x - 3)(x + 3).$$

ILLUSTRATION 5. To factor $25x^2 - 9y^4$, we observe that $25x^2 = (5x)^2$ and $9y^4 = (3y^2)^2$. Hence,

$$25x^2 - 9y^4 = (5x - 3y^2)(5x + 3y^2).$$

ILLUSTRATION 6. $$a^4 - 16y^4 = (a^2 - 4y^2)(a^2 + 4y^2) = (a - 2y)(a + 2y)(a^2 + 4y^2).$$

74. Perfect square trinomials *

An integral rational polynomial with three terms is called a *trinomial.* The square of any binomial is a *perfect square trinomial.* A trinomial of this type can be recognized and factored by the formulas of Types III and IV of Section 70.

Perfect square trinomials:

III. $$a^2 + 2ab + b^2 = (a + b)^2;$$

IV. $$a^2 - 2ab + b^2 = (a - b)^2.$$

In a perfect square trinomial, we notice that

1. *two terms are perfect squares, and*
2. *the third term is plus (or minus) twice the product of the square roots of the other terms.*

To verify that a trinomial is a perfect square, take the square roots of the terms which are perfect squares, compute the third term which should be present, and check by inspection.

ILLUSTRATION 1. To factor $4x^2 - 20xy + 25y^2$, we observe perfect squares $4x^2$ and $25y^2$, whose square roots are $2x$ and $5y$. Hence the third term should be $-2(2x)(5y) = -20xy$, which checks, and gives

$$4x^2 - 20xy + 25y^2 = (2x - 5y)^2.$$

ILLUSTRATION 2. $$16z^4 + 24z^2w + 9w^2 = (4z^2 + 3w)^2.$$

EXERCISE 31

Factor by use of Types I *and* II. *If fractions occur, leave the factors in the form which arises most naturally by standard methods. Check by multiplying the factors.*

1. $3x + bx$.
2. $2cx + 4dx$.
3. $6xy^2 + 2ax$.
4. $bx + x + c^2x$.
5. $2cy + d^2y + y$.
6. $3ab + 2a - 5a^2$.
7. $-ax + 3bx + cx$
8. $-5y^2 - 3y^3 + ay^2$.
9. $-4at + t^2 - ct^3$.
10. $4b^2x^3 + 6bx^2 + 8bcx^2$.
11. $3a^2y^3 - 2ay^2 + a^3y^4$.

* See Note 5 in the Appendix for an explanation of the process for finding the square root of a number expressed in decimal notation, by means of pure arithmetic. This process is intimately related to the formula of Type III.

12. $5a^3u^3 - 3au^2 + 4a^3u$.
13. $2w^4x^3 - 6w^2x^2 + 5w^3x^4$.

14. $x^2 - a^2$.
15. $w^2 - z^2$.
16. $y^2 - 25$.

17. $64 - x^2y^2$.
18. $36 - z^2$.
19. $4x^2 - y^2$.

20. $9x^2 - 25z^2$.
21. $36d^2 - 121$.
22. $9z^2 - 1$.

23. $4a^2 - 9b^2$.
24. $1 - 25x^2$.
25. $256a^2 - 1$.

26. $9z^2 - \frac{1}{4}$.
27. $\frac{1}{9} - w^2$.
28. $9a^2b^2w - 16z^2w$.

29. $25w^2 - c^2d^2$.
30. $49u^2 - 16v^2w^2$.
31. $36a^2b^2 - 64x^2$.

32. $9z^2 - 144a^2b^2$.
33. $ax^4 - ay^4$.
34. $16y - x^4y$.

Which trinomials are not perfect squares?

35. $x^2 + 3x + 4$.
36. $a^2 + a + 1$.

37. $4x^2 + 6x + 9$.
38. $9x^4 - 6x^2 + 1$.

39. $3x^2 + 6xy + 4y^2$.
40. $4z^2 - 12wz + 9w^2$.

Insert any missing term to complete a perfect square. Then factor by use of Types III *and* IV.

41. $x^2 + 2bx + b^2$.
42. $d^2 + 2dy + y^2$.

43. $a^2 - 2a + 1$.
44. $x^2 + (\quad) + 16$.

45. $w^2 - (\quad) + 36$.
46. $4x^2 - 20xz + 25z^2$.

47. $x^2 + 81 - 18x$.
48. $x^2 + x + \frac{1}{4}$.

49. $49x^2 + 14ax + a^2$.
50. $1 + x^2z^2 - 2xz$.

51. $64 - 16ab + a^2b^2$.
52. $9a^2 + (\quad) + 25b^2$.

53. $4x^2 + (\quad) + 9z^2$.
54. $16b^2 - (\quad) + 49x^2$.

55. $4c^2d^2 - (\quad) + 25a^2$.
56. $9x^2 - (\quad) + h^2w^2$.

57. $-30xy + 9x^2 + 25y^2$.
58. $24ax + 9x^2 + 16a^2$.

59. $4x^4 - 28x^2 + 49$.
60. $25 - 30x^2 + 9x^4$.

61. $4a^4 - 12a^2b^2 + 9b^4$.
62. $4u^6 + 12u^3v^3 + 9v^6$.

Factor.

63. $49z^2 - 4b^2$.
64. $75a^2 - 3a^2b^2$.
65. $36uv^2 - 4uw^2$.

66. $25z^2 - 30uz + 9u^2$.
67. $x^2 + 25y^4 - 10xy^2$.

68. $9x^4 + 49y^4 - 42x^2y^2$.
69. $4a^2x - 4ax + x$.

70. $x^4 - 9y^4$. **71.** $98u^4 - 50$. **72.** $9ax^2 - 4ay^4$.

73. $25x^2 - 100b^4$. **74.** $3a^2x^2 - 5a^2y^4$. **75.** $16x^4 - 625v^4$.

76. $25u^4v^4 - 70u^2v^2w^2 + 49w^4$. **77.** $18u^2 - 60uv + 50v^2$.

78. $2u^6 - 12u^3v^4 + 18v^8$. **79.** $147x^2 - 75v^4w^4$.

First factor and then compute. Check by expanding the original expression.

80. $23^2 - 17^2$. **81.** $52^2 - 48^2$. **82.** $27^2 - 23^2$.

83. $104^2 - 96^2$. **84.** $45^2 - 55^2$. **85.** $37^2 - 33^2$.

75. Factoring trinomials by a trial and error method

We recall the formulas of Types V and VI of Section 70.

V. $\boldsymbol{x^2 + (a + b)x + ab = (x + a)(x + b).}$

VI. $\boldsymbol{acx^2 + (ad + bc)x + bd = (ax + b)(cx + d).}$

Certain trinomials of the form * $gx^2 + hx + k$ can be factored by a trial and error method suggested by the preceding formulas.

Example 1. Factor: $x^2 - 2x - 8$.

Solution. 1. We wish to find a and b so that

$$(x + a)(x + b) = x^2 + (a + b)x + ab = x^2 - 2x - 8.$$

2. Hence, $ab = -8$; thus a and b have opposite signs and are factors of 8. Since the sum of the cross products is $-2x$, we guess that $a = -4$ and $b = 2$. This is correct because

$$(x - 4)(x + 2) = x^2 - 2x - 8.$$

Example 2. Factor: $15x^2 + 2x - 8$.

Solution. 1. We wish to find a, b, c, and d so that

$$(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd = 15x^2 + 2x - 8.$$

Hence, $ac = 15$, $bd = -8$, and the sum of the cross products is $2x$.

2. *First trial.* Since $ac = 15$, choose $a = 15$ and $c = 1$; since $bd = -8$, choose $b = 2$ and $d = -4$. This selection is wrong because

$$(15x + 2)(x - 4) = 15x^2 - 58x - 8.$$

3. *Second trial.* Choose $a = 3, c = 5, b = -2$, and $d = 4$. This selection is correct because

$$(3x - 2)(5x + 4) = 15x^2 + 2x - 8.$$

* If g, h, and k were chosen at random, without a common factor, the trinomial would probably be prime. Later, we shall discuss a condition which g, h, and k satisfy when and only when the trinomial is *not* prime.

If one prime factor is merely the *negative* of another, we do *not* consider them as distinct prime factors; we combine their powers into a single power of one of them.

ILLUSTRATION 1. In $(-x-2)(x+2) = -x^2 - 4x - 4$, we notice that $(-x-2) = -(x+2)$. Hence, we write

$$-x^2 - 4x - 4 = -(x+2)(x+2) = -(x+2)^2.$$

Note 1. The preceding factoring methods apply to polynomials in which the coefficients are any real numbers, not merely integers as in the illustrations. The nature of the coefficients which we agree to allow in a polynomial and its factors affects our definition of a *prime* expression but not our general factoring procedure.

EXAMPLE 3. Factor: $6x^4 - x^2 - 15.$

SOLUTION. By trial and error, $6x^4 - x^2 - 15 = (3x^2 - 5)(2x^2 + 3).$

EXERCISE 32

Factor by trial and error methods.

1. $x^2 + 8x + 15.$ **2.** $x^2 + 10x + 21.$

3. $a^2 - 8a + 12.$ **4.** $y^2 - 7y + 12.$

5. $x^2 - 8x + 15.$ **6.** $z^2 - 5z - 6.$

7. $t^2 + 4t - 21.$ **8.** $w^2 - 5w - 24.$

9. $x^2 - 3x - 18.$ **10.** $a^2 + 6a - 16.$

11. $w^2 + 2w - 48.$ **12.** $4 - 3y - y^2.$

13. $15 - 2w - w^2.$ **14.** $8 - 7a - a^2.$

15. $24 + 2w - w^2.$ **16.** $b^2 + 3b - 28.$

17. $32 - 4y - y^2.$ **18.** $27 + 6w - w^2.$

19. $54 - 3k - k^2.$ **20.** $36 + 5h - h^2.$

21. $x^2 - 6x - 72.$ **22.** $2x^2 + 7x + 3.$

23. $5a^2 + 12a + 7.$ **24.** $3a^2 + 8a + 5.$

25. $10x^2 - 11x + 3.$ **26.** $3a^2 - 10a + 7.$

27. $8x^4 - 10x^3 + 3x^2.$ **28.** $2x^4 - x^3 - 6x^2.$

29. $3y^3 + 2y^2 - 5y.$ **30.** $3h^3 + 4h^2 - 7h.$

31. $3x^6 + x^3 - 10$.
32. $15y^2 + 4y - 4$.
33. $8w^6 - 6w^3 - 9$.
34. $-5 + 3x + 2x^2$.
35. $15a^4 - a^2 - 28$.
36. $8 + 2y^2 - 15y^4$.
37. $7 - 19x - 6x^2$.
38. $-12h^2 - 8h + 15$.
39. $-27x^2 + 3x + 2$.
40. $5a^2 + 12ab + 7b^2$.
41. $3x^2 + 5xy + 2y^2$.
42. $3x^2 + 7ax - 6a^2$.
43. $8w^2 + 14wz - 15z^2$.
44. $18w^4 + 9w^2 - 20$.
45. $-5u^2 - 28uw + 12w^2$.
46. $45x^2 - 8xy - 4y^2$.

Factor by the appropriate method.

47. $6a^2 - 13ab + 5b^2$.
48. $4x^2 - 7xy + 3y^2$.
49. $100a^2 - x^4$.
50. $49z^2 - 4b^2$.
51. $x^4 - 16y^4$.
52. $7c^2 + 19cd - 6d^2$.
53. $64a^2 - 48ac + 9c^2$.
54. $-2x^2 + 15 + x$.
55. $-6x^2 + 20 - 7x$.
56. $9 + 25v^2w^2 - 30vw$.
57. $2x^3 - 2xy^2$.
58. $8a^2c - 18c^3$.
59. $9x^4 + 12x^2y + 4y^2$.
60. $3a + 13ab + 10ab^2$.
61. $25x^2 - 100b^4$.
62. $75cd^2 + 30c^2d + 3c^3$.
63. $2r - 11hr + 15h^2r$.
64. $.81c^4 - .16d^4$.
65. $\frac{1}{81} - 16y^4$.
66. $5x^4 - 16x^2 + 3$.
67. $31x - 5x^2 - 3$.
68. $x^2y^2 + 9xy - 52$.
69. $3x^4 - 7x^2 - 20$.
70. $6x - 1 - 9x^2$.
71. $625w^4z^4 - 16z^4$.
72. $9x^{2k} - 4$.
73. $4y^{2h} - z^{2n}$.
74. $-4x^2 + 12x - 9$.
75. $-9a^2 + 30ab - 25b^2$.
76. $3x^4 - 17x^2 + 10$.
77. $2x^4 + x^2 - 15$.
78. $3x^4 - 5x^2y^2 - 2y^4$.
79. $3a^4 + 4x^2y^2 - 15y^4$.

76. Factoring by use of grouping

The following methods make frequent use of the fact that an expression enclosed within parentheses should be treated as a single number expression.

ILLUSTRATION 1. $5(x-a)-3(x-a)=2(x-a)$.

ILLUSTRATION 2. To factor the following expression, we observe the common factor $(a-b)$, and remove it from each term:

$$2c(a-b)+d(a-b)=(a-b)(2c+d).$$

ILLUSTRATION 3. After grouping, we observe a common binomial factor, and then complete the factoring:

$$bx+by+2hx+2hy=(bx+by)+(2hx+2hy)$$
$$=b(x+y)+2h(x+y)=(b+2h)(x+y).$$

ILLUSTRATION 4. The second term below was altered by changing signs (or, multiplying by -1) both within and without the parentheses in order to exhibit the same binomial factor as the first term:

$$3x(2a-b)+4y(b-2a)=3x(2a-b)-4y(2a-b)$$
$$=(3x-4y)(2a-b).$$

ILLUSTRATION 5. In order to factor below, we group two terms within parentheses preceded by a minus sign, and hence change the signs of the terms, in order to exhibit the same factor as observed in the other terms:

$$xz-kx+kw-wz=(xz-wz)-(kx-kw)$$
$$=z(x-w)-k(x-w)=(z-k)(x-w).$$

ILLUSTRATION 6. $6-3x^2-8x+4x^3=(6-8x)-(3x^2-4x^3)$
$$=2(3-4x)-x^2(3-4x)=(3-4x)(2-x^2).$$

ILLUSTRATION 7. We factor below as the difference of two squares:

$$(c-2x)^2-(b-a)^2=[(c-2x)-(b-a)][(c-2x)+(b-a)]$$
$$=(c-2x-b+a)(c-2x+b-a).$$

ILLUSTRATION 8. $a^2-c^2+b^2-d^2-2ab-2cd$
$$=(a^2-2ab+b^2)-(c^2+2cd+d^2)=(a-b)^2-(c+d)^2$$
$$=[(a-b)-(c+d)][(a-b)+(c+d)]$$
$$=(a-b-c-d)(a-b+c+d).$$

EXERCISE 33

Factor.

1. $7(x+2y)-5(x+2y)$.
2. $4(3h+k)-9(3h+k)$.
3. $c(x+y)+d(x+y)$.
4. $5a(c-3d)-2b(c-3d)$.
5. $2h(m-2)-3k(m-2)$.
6. $2c(x+4y)+3d(x+4y)$.

7. $-5c(r+s)+2d(r+s)$.
8. $-2x(a+h)-3y(a+h)$.
9. $3h(w-z)-(w-z)$.
10. $2x(h-2k)+3hy-6ky$.
11. $3a(w-2k)+2bw-4bk$.
12. $bx+by+2hx+2hy$.
13. $3ac+3bc+ad+bd$.
14. $2ax+2ay+bx+by$.
15. $cr-cs+3dr-3ds$.
16. $4hx-4bh+5cx-5bc$.
17. $2cx+cy-2dx-dy$.
18. $5ax+2bx-10ad-4bd$.
19. $4hx-4bh-8cx+8bc$.
20. $3bw-3bz-4aw+4az$.
21. $(x^3-2x^2)-(x-2)$.
22. $(ax^3+bx^2)-4(ax+b)$.
23. x^3+2x^2+x+2.
24. $ax^2+bx^2+ad^2+bd^2$.
25. x^3-3x^2+x-3.
26. $2x^2-4x+1-8x^3$.
27. a^3-3a^2-3+a.
28. $2+4x-10x^4-5x^3$.
29. $3x^3-2x^2+6x-4$.
30. $4-8x^2-5x+10x^3$.
31. $2(r-s)-x(s-r)$.
32. $a(x-y)+b(y-x)$.
33. $x^2-(s+3)^2$.
34. $(w-1)^2-16k^2$.
35. $(2z+w)^2-y^2$.
36. $(4a-b)^2-(2x-y)^2$.
37. $(c-3d)^2-(2x+y)^2$.
38. $(4x-3y)^2-25$.
39. $z^2+2z+1-9x^2$.
40. $4w^2+20w+25-81z^2$.
41. $y^2+2yz+z^2-4x^2$.
42. $9w^2-4a^2-4ab-b^2$.
43. $4a^2-9z^2-6z-1$.
44. $16y^2-a^2+2ab-b^2$.
45. $9x^2-y^2+2yz-z^2$.
46. $w^2-4x^2-y^2-4xy$.
47. $16a^2-1-9x^2+6x$.
48. $a^2c-a^2d+b^2d-b^2c$.
49. $bx^4-by^4+cx^4-cy^4$.
50. a^2-b^2-a+b.
51. $z^4-w^2+w-z^2$.
52. $ch+6dk+3dh+2ck$.
53. $r^2+6rt+9t^2-a^2-2ab-b^2$.
54. $4x^2+4xy+y^2-9a^2-12at-4t^2$.
55. $c^2+4c+4-9d^2-6dh-h^2$.
56. $16x^2-24xy+9y^2-9a^2-12a-4$.
57. $9x^2-6xy+y^2-25a^2+10ab-b^2$.

58. $4x^2 - 4xy + y^2 - 9z^2 + 6wz - w^2$.

59. $b^2 - 9x^2 + 2ab + a^2$.

60. $4d^2 - 16w^2 - 4cd + c^2$.

61. $4a^2 + 9b^2 - 4x^2 - y^2 - 4xy - 12ab$.

62. $a^2 - 9b^2 - d^2 - 2a - 1 - 6bd$.

63. $16x^4 - 81y^4 + 4x^2 - 9y^2$.

64. $c^2x^4 - 81c^2 + 324 - 4x^4$.

77. Cube of a binomial

We verify that

$$(x + y)^3 = (x + y)^2(x + y) = (x^2 + 2xy + y^2)(x + y)$$
$$= x^3 + 2x^2y + xy^2 + x^2y + 2xy^2 + y^3.$$

On collecting terms we obtain (1) and, similarly, we could verify (2):

$$\mathbf{(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3;} \qquad (1)$$
$$\mathbf{(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3.} \qquad (2)$$

The student should memorize these formulas.

ILLUSTRATION 1. From formula 1, with $x = 2a$ and $y = b$,

$$(2a + b)^3 = (2a)^3 + 3(2a)^2(b) + 3(2a)(b^2) + b^3$$
$$= 8a^3 + 12a^2b + 6ab^2 + b^3.$$

ILLUSTRATION 2. From formula 2,

$$(4 - x)^3 = 64 - 48x + 12x^2 - x^3.$$

78. Sum and difference of two cubes

By long division we could verify that

$$\frac{a^3 - b^3}{a - b} = a^2 + ab + b^2; \qquad \frac{a^3 + b^3}{a + b} = a^2 - ab + b^2.$$

Hence, we have the following formulas, useful for factoring when read from *left to right,* and useful in multiplication when read from *right to left.*

$$\mathbf{a^3 - b^3 = (a - b)(a^2 + ab + b^2)} \qquad (1)$$
$$\mathbf{a^3 + b^3 = (a + b)(a^2 - ab + b^2)} \qquad (2)$$

ILLUSTRATION 1. By use of (1), read from right to left, with $b = 3$,

$$(a - 3)(a^2 + 3a + 9) = a^3 - 3^3 = a^3 - 27.$$

ILLUSTRATION 2. From formula 2 with $a = 3x$ and $b = 2y$,

$$\begin{aligned} 27x^3 + 8y^3 &= (3x)^3 + (2y)^3 \\ &= (3x + 2y)[(3x)^2 - (3x)(2y) + (2y)^2] \\ &= (3x + 2y)(9x^2 - 6xy + 4y^2). \end{aligned}$$

ILLUSTRATION 3. $1 - 64x^3 = 1^3 - (4x)^3 = (1 - 4x)(1 + 4x + 16x^2).$

EXAMPLE 1. Factor: $y^6 - 19y^3 - 216.$

SOLUTION. $$y^6 - 19y^3 - 216 = (y^3 - 27)(y^3 + 8)$$
$$= (y - 3)(y^2 + 3y + 9)(y + 2)(y^2 - 2y + 4).$$

EXERCISE 34

Divide by long division and check by use of Section 78.

1. $\dfrac{x^3 + y^3}{x + y}$. 2. $\dfrac{a^3 - h^3}{a - h}$. 3. $\dfrac{a^3 + 27b^3}{a + 3b}$. 4. $\dfrac{8x^3 - 27y^3}{2x - 3y}$.

Multiply by inspection.

5. $(c + w)(c^2 - cw + w^2)$. 6. $(u - v)(u^2 + uv + v^2)$.

7. $(3a - c)(9a^2 + 3ac + c^2)$. 8. $(1 - w)(1 + w + w^2)$.

9. $(1 - 3x)(1 + 3x + 9x^2)$. 10. $(2 - 3u)(4 + 6u + 9u^2)$.

11. $(b - 2x)(b^2 + 2bx + 4x^2)$. 12. $(4y + 1)(16y^2 - 4y + 1)$.

Factor.

13. $d^3 - y^3$. 14. $h^3 + z^3$. 15. $y^3 - 27$. 16. $u^3 + 1$.

17. $1 - v^3$. 18. $8 - x^3$. 19. $z^3 + 1000$. 20. $64 - w^3$.

21. $1 - 27x^3$. 22. $125 + 8y^3$. 23. $z^3 - 8w^3$. 24. $8 - 27x^3$.

25. $216x^3 - y^3z^3$. 26. $x^3 - 64y^3$. 27. $343a^3 - 8x^3z^3$.

Expand each cube by use of the formulas of Section 77.

28. $(c + d)^3$. 29. $(h - k)^3$. 30. $(2 + y)^3$. 31. $(u + 3)^3$.

32. $(5 - y)^3$. 33. $(2x + w)^3$. 34. $(y - 3x)^3$. 35. $(4x + y)^3$.

36. $(a - b^2)^3$. 37. $(a^2 - 2x)^3$. 38. $(x^2 - y^2)^3$. 39. $(c - 2b^2)^3$.

40. $(a - 2z^3)^3$. 41. $(2c^2 - 3z)^3$. 42. $(.1 - 2x)^3$.

Factor.

43. $x^6 + 7x^3 - 8.$

44. $27b^6 + 26b^3 - 1.$

45. $8x^6 - 19x^3y^3 - 27y^6.$

46. $64a^6 - 16a^3b^3 + b^6.$

47. $a^3 - 3a^2 + 3a - 1.$

48. $z^3 + 6z^2 + 12z + 8.$

49. $w^3 - 9w^2x + 27wx^2 - 27x^3.$

50. $125u^3 - 75u^2 + 15u - 1.$

51. $(c - d)^3 - a^3.$

52. $(h - x)^3 - (y - x)^3.$

★79. Trinomials equal to differences of squares

An expression of the form $x^4 + kx^2y^2 + y^4$ can be written as the difference of two squares if the expression becomes a perfect square after the *addition* of a *perfect square multiple* of x^2y^2.

EXAMPLE 1. Factor: $\qquad 64a^4 - 64a^2b^2 + 25b^4.$

SOLUTION. 1. A perfect square involving $64a^4$ and $25b^4$ is

$$(8a^2 + 5b^2)^2 = 64a^4 + 80a^2b^2 + 25b^4.$$

2. Hence, $64a^4 - 64a^2b^2 + 25b^4$ becomes a perfect square if we add $144a^2b^2$. Therefore, we add $144a^2b^2$ and, to compensate for this, also subtract $144a^2b^2$:

$$64a^4 - 64a^2b^2 + 25b^4 = (64a^4 - 64a^2b^2 + 25b^4 + 144a^2b^2) - 144a^2b^2$$
$$= (64a^4 + 80a^2b^2 + 25b^4) - 144a^2b^2 = (8a^2 + 5b^2)^2 - 144a^2b^2$$
$$= (8a^2 + 5b^2 - 12ab)(8a^2 + 5b^2 + 12ab).$$

EXAMPLE 2. Factor: $\qquad 9x^4 - 16x^2y^2 + 4y^4. \qquad (1)$

SOLUTION. 1. The perfect squares involving $9x^4$ and $4y^4$ are

$$(3x^2 \pm 2y^2)^2 = 9x^4 \pm 12x^2y^2 + 4y^4.$$

In order to obtain $+ 12x^2y^2$ from (1), we would have to add $28x^2y^2$, but this is not a perfect square. To obtain $- 12x^2y^2$ we must add $4x^2y^2$, which is a perfect square.

2. Add, and also subtract, $4x^2y^2$ in (1):

$$9x^4 - 16x^2y^2 + 4y^4 = (9x^4 - 16x^2y^2 + 4y^4 + 4x^2y^2) - 4x^2y^2$$
$$= (3x^2 - 2y^2)^2 - 4x^2y^2 = (3x^2 - 2y^2 - 2xy)(3x^2 - 2y^2 + 2xy).$$

★EXERCISE 35

Factor by reducing to a difference of two squares.

1. $a^4 + a^2 + 1.$

2. $z^4 - 3z^2 + 1.$

3. $9a^4 + 2a^2 + 1.$

4. $9x^4 + 11x^2 + 4.$

5. $z^4 + h^2z^2 + h^4.$

6. $9x^4 - 10x^2 + 1.$

7. $4w^4 + 8a^2w^2 + 9a^4$. **8.** $a^4 - 9a^2y^2 + 16y^4$. **9.** $25a^4 - 5a^2b^2 + 4b^4$.

10. $4d^4 + 4d^2h^2 + 25h^4$. **11.** $x^4 + 4$. **12.** $w^4 + 4x^4$.

13. $z^4 + 64h^4$. **14.** $625x^4 + 4w^4$. **15.** $81z^4 + 64x^4$.

16. $x^4 - 12a^2x^2 + 16a^4$. **17.** $9a^4 - 16a^2c^2 + 4c^4$.

18. $19a^2x^2 + 4x^4 + 49a^4$. **19.** $25a^4 + 9y^4 - 34a^2y^2$.

20. $4x^4 - 24x^2y^2 + 25y^4$. **21.** $9x^4 - 39x^2y^2 + 25y^4$.

★80. Perfect powers

An integral rational term is said to be a *perfect nth power* if it is the nth power of an integral rational term.

ILLUSTRATION 1. $16a^4b^8$ is a perfect 4th power because $16a^4b^8 = (2ab^2)^4$.

ILLUSTRATION 2. $8a^6b^6$ is a perfect cube because $8a^6b^6 = (2a^2b^2)^3$. The original exponents have 3 as a factor.

In a perfect nth power, each exponent has n as a factor because in raising a term to the nth power we multiply each of the original exponents by n.

ILLUSTRATION 3. $$(2^3a^2b^4)^n = 2^{3n}a^{2n}b^{4n}.$$

★81. Special cases of sum or difference of perfect powers

I. *If n is even, commence factoring $(a^n - b^n)$ by recognizing it as the difference of two squares.*

ILLUSTRATION 1.
$$\begin{aligned} x^6 - y^6 &= (x^3)^2 - (y^3)^2 = (x^3 - y^3)(x^3 + y^3) \\ &= (x - y)(x^2 + xy + y^2)(x + y)(x^2 - xy + y^2). \end{aligned}$$

We could have commenced by factoring $(x^6 - y^6)$ as the difference of two *cubes*, but this would have been an inefficient method.

ILLUSTRATION 2. To factor $16a^4b^4 - 81$, we observe that each term is a perfect square. Hence,
$$\begin{aligned} 16a^4b^4 - 81 &= (4a^2b^2 - 9)(4a^2b^2 + 9) \\ &= (2ab - 3)(2ab + 3)(4a^2b^2 + 9), \end{aligned}$$
where the final factor is a prime sum of perfect squares.

II. *If n is odd and has 3 as a factor, we can commence factoring $(a^n \pm b^n)$ by recognizing it as the sum or difference of two cubes.*

ILLUSTRATION 3. $x^9 + y^9 = (x^3)^3 + (y^3)^3$
$= (x^3 + y^3)(x^6 - x^3y^3 + y^6)$ (Using Section 78)
$= (x + y)(x^2 - xy + y^2)(x^6 - x^3y^3 + y^6).$

★EXERCISE 36

Express the perfect power as the 3d *or* 4th *power of some other term, whichever is the case.*

1. $8a^3b^3$. **2.** $27a^6y^9$. **3.** $16a^4b^4$. **4.** $81x^8y^4$.

5. $125x^6y^3$. **6.** $64x^9y^9$. **7.** $256u^8v^{12}$. **8.** $625a^4y^8$.

Factor each expression which is not prime.

9. $a^4 - x^4$. **10.** $y^4 - 81$. **11.** $16 - w^4$. **12.** $81x^4 - y^4$.

13. $x^8 - y^8$. **14.** $x^4 + y^4$. **15.** $81 - 16x^4$. **16.** $y^6 - x^6$.

17. $u^6 - 1$. **18.** $a^6 - 64$. **19.** $x^6 - 64y^6$. **20.** $a^6 + 64$.

21. $x^6 + 1$. **22.** $729 - a^6$. **23.** $729 + x^6$. **24.** $125 - a^6$.

25. $256 - a^8$. **26.** $h^9 - k^9$. **27.** $a^9 + b^9$. **28.** $a^8 + b^8$.

29. $81x^8 - y^4$. **30.** $16x^4 - 81y^8$. **31.** $625 - 16x^8$. **32.** $x^8 - u^8v^8$.

33. $a^6 - 64b^6$. **34.** $64 + x^6y^6$. **35.** $8a^3 - 27x^6$. **36.** $x^{16} - y^{16}$.

★82. Properties of factors of $a^n \pm b^n$

We have verified special cases of the following results, where n represents a positive integer. Any special case of the results can be checked * by long division.

I. *For every value of* n, $(a^n - b^n)$ *has* $(a - b)$ *as a factor; in other words,* $(a^n - b^n)$ *is exactly divisible by* $(a - b)$.

ILLUSTRATION 1. $a^3 - b^3 = (a - b)(a^2 + ab + b^2).$
$a^4 - b^4 = (a - b)(a^3 + a^2b + ab^2 + b^3).$
$a^4 - 16 = a^4 - 2^4 = (a - 2)(a^3 + 2a^2 + 4a + 8).$

II. *If* n *is even,* $(a^n - b^n)$ *has* $(a + b)$ *as a factor.*

ILLUSTRATION 2. $a^2 - b^2 = (a - b)(a + b).$
$a^4 - b^4 = (a + b)(a^3 - a^2b + ab^2 - b^3).$

* A convenient method for giving a general proof of the results is met in a more advanced section of algebra.

III. *If n is odd,* $(a^n + b^n)$ *has* $(a + b)$ *as a factor.*

ILLUSTRATION 3. $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.
$$a^7 + b^7 = (a + b)(a^6 - a^5b + a^4b^2 - a^3b^3 + a^2b^4 - ab^5 + b^6).$$

IV. *If n is even,* $(a^n + b^n)$ *does not have either* $(a - b)$ *or* $(a + b)$ *as a factor.*

ILLUSTRATION 4. $(a^2 + b^2)$ and $(a^4 + b^4)$ are prime. $(a^6 + b^6)$ is not prime but it does not have either $(a + b)$ or $(a - b)$ as a factor:
$$a^6 + b^6 = (a^2 + b^2)(a^4 - a^2b^2 + b^4),$$
where each factor is prime.

Special cases of the following general properties were exhibited by the second factors in Illustrations 1, 2, and 3.

A. *When* $(a^n - b^n)$ *is divided by* $(a - b)$, *all coefficients in the quotient are* $+1$.

B. *When* $(a^n + b^n)$ *or* $(a^n - b^n)$ *is divided by* $(a + b)$, *the coefficients in the quotient are alternately* $+1$ *and* -1.

Factors obtained by reference to (I), (II), and (III) are not always prime. Also, as seen in Illustration 4 and Section 81, an expression of the type $a^n + b^n$, with *n even,* may be factorable although (IV) is true. In finding the prime factors of $a^n \pm b^n$, first use the methods of Section 81 if possible, before employing (I), (II), and (III).

ILLUSTRATION 5. $x^6 + 64 = (x^2)^3 + 4^3 = (x^2 + 4)(x^4 - 4x^2 + 16)$.

ILLUSTRATION 6.
$$\begin{aligned} x^9 - y^9 &= (x^3)^3 - (y^3)^3 = (x^3 - y^3)(x^6 + x^3y^3 + y^6) \\ &= (x - y)(x^2 + xy + y^2)(x^6 + x^3y^3 + y^6). \end{aligned}$$

ILLUSTRATION 7.
$$\begin{aligned} x^4 - y^4 &= (x^2)^2 - (y^2)^2 = (x^2 - y^2)(x^2 + y^2) \\ &= (x - y)(x + y)(x^2 + y^2). \qquad (1) \end{aligned}$$
By use of (I),
$$x^4 - y^4 = (x - y)(x^3 + x^2y + xy^2 + y^3). \qquad (2)$$
Equation 1 shows that the second factor in (2) is not prime; this factor could be factored by grouping:
$$\begin{aligned} x^3 + x^2y + xy^2 + y^3 &= x^2(x + y) + y^2(x + y) \\ &= (x + y)(x^2 + y^2). \end{aligned}$$
Thus, we finally arrive at the factors obtained in (1) but by a much less desirable process.

★EXERCISE 37

Find the quotient by long division, and the remainder if the division is inexact.

1. $\dfrac{x^6 - y^6}{x - y}$. 2. $\dfrac{x^7 + y^7}{x + y}$. 3. $\dfrac{x^5 + 32y^5}{x + 2y}$. 4. $\dfrac{x^4 + 16}{x + 2}$.

Find each result without using long division, by use of properties A *and* B *of Section* 82, *and check by multiplication.*

5. $\dfrac{a^5 + y^5}{a + y}$. 6. $\dfrac{u^8 - v^8}{u - v}$. 7. $\dfrac{x^8 - y^8}{x + y}$. 8. $\dfrac{z^5 + 1}{z + 1}$.

9. $\dfrac{z^5 - w^5}{z - w}$. 10. $\dfrac{x^7 - y^7}{x - y}$. 11. $\dfrac{x^5 - 1}{x - 1}$. 12. $\dfrac{u^4 - 16}{u + 2}$.

13. $(x^{10} - y^{10}) \div (x + y)$. 14. $(16x^4 - a^4) \div (2x + a)$.

15. $(a^3 - 8) \div (a - 2)$. 16. $(243x^5 - 1) \div (3x - 1)$.

17. $\dfrac{x^8 - y^8}{x^4 - y^4}$. 18. $\dfrac{a^9 + b^9}{a^3 + b^3}$. 19. $\dfrac{a^8 - b^4}{a^2 + b}$. 20. $\dfrac{x^{10} - y^{10}}{x^2 - y^2}$.

21. $\dfrac{x^9 - y^9}{x^3 - y^3}$. 22. $\dfrac{y^6 - x^3}{y^2 - x}$. 23. $\dfrac{x^{12} - y^{12}}{x^3 + y^3}$. 24. $\dfrac{x^{10} - 32a^5}{x^2 - 2a}$.

25. $\dfrac{8x^3 - 27}{2x - 3}$. 26. $\dfrac{a^8 - 16b^4}{a^2 + 2b}$. 27. $\dfrac{z^5 - 32x^{10}}{z - 2x^2}$.

Factor each expression which is not prime.

28. $a^5 - c^5$. 29. $a^4 - w^4$. 30. $u^7 - v^7$. 31. $u^5 + w^5$.

32. $32 + x^5$. 33. $1 - y^6$. 34. $x^8 - 256y^8$. 35. $u^9 - v^9$.

36. $v^5 - 32u^5$. 37. $32a^5 - 1$. 38. $x^{12} + y^{12}$. 39. $128 + x^7$.

40. $x^5 - 243y^5$. 41. $a^3 - 27x^6$. 42. $16x^4 + y^4$.

43. $4x^4 + 1$. 44. $4w^4x^4 + 81z^4$. 45. $u^{3h} + v^3$.

46. $32x^{5k} + y^5$. 47. $x^{16} + y^{16}$. 48. $u^{6h} - v^{6k}$.

49. $512 - x^9$. 50. $x^9 + 512a^9$. 51. $u^{3h} + v^{3k}$.

CHAPTER *6*

ADVANCED TOPICS IN FRACTIONS

83. Reduction of fractions to lowest terms

Whenever we make a reference to *factoring* in a fraction, it will be assumed that the numerator and denominator are integral rational polynomials with integral coefficients. In the final result of any operation on fractions, we agree to leave any expression in a factored form if it arises naturally.

SUMMARY. *To reduce a fraction to lowest terms:*

1. *Factor the numerator and denominator.*

2. *Divide both numerator and denominator by all their common factors.*

ILLUSTRATION 1. In the following fraction, we divide both numerator and denominator by $3x - 4y$ and indicate this by cancellation.

$$\frac{9x^2 - 16y^2}{3x^2 + 2xy - 8y^2} = \frac{\cancel{(3x - 4y)}(3x + 4y)}{\cancel{(3x - 4y)}(x + 2y)} = \frac{3x + 4y}{x + 2y}.$$

ILLUSTRATION 2. In reducing the following fraction to lowest terms, we first notice that one factor in the numerator is merely the negative of a factor of the denominator.

$$\frac{x^2 - 9}{12 + 2x - 2x^2} = \frac{(x - 3)(x + 3)}{2(3 - x)(2 + x)}$$

$$= -\frac{\cancel{(x - 3)}(x + 3)}{2\cancel{(x - 3)}(2 + x)} = -\frac{x + 3}{2(x + 2)}. \qquad \text{(Divide out } x - 3\text{)}$$

In the preceding line, we obtained $(x - 3)$ in the denominator by multiplying $(3 - x)$ by $- 1$, and hence it was necessary to change the sign before the fraction to keep its value unaltered.

EXERCISE 38

Reduce to lowest terms.

1. $\dfrac{75x^3y^4}{50x^2y^6}$.
2. $\dfrac{a^3b(x-2y)}{a^2b^4(x-2y)}$.
3. $\dfrac{5c+5d}{3c+3d}$.
4. $\dfrac{ax(x+y)}{cx(x+y)}$.
5. $\dfrac{c^2d(a+3b)}{cd^3(a+3b)}$.
6. $\dfrac{4ay-2by}{2ax-bx}$.
7. $\dfrac{x^2-y^2}{2x-2y}$.
8. $\dfrac{ax-cx}{a^2-c^2}$.
9. $\dfrac{5x^2-20}{2x-4}$.
10. $\dfrac{4x^2-4y^2}{cx+cy}$.
11. $\dfrac{4a^2-9b^2}{2ax-3bx}$.
12. $\dfrac{4a^2-b^2}{4ax-2bx}$.
13. $\dfrac{m^2-m-42}{m^2-3m-28}$.
14. $\dfrac{a^2+2a-15}{a^2+a-20}$.
15. $\dfrac{3x^2+13x-10}{9x^2-4}$.
16. $\dfrac{3x^2-7ax+4a^2}{3x^2+2ax-8a^2}$.
17. $\dfrac{2x^2+2ax-12a^2}{x^2+6ax+9a^2}$.
18. $\dfrac{a^2-4ax-5x^2}{2a^2-9ax-5x^2}$.
19. $\dfrac{2x-4y}{x^2-4xy+4y^2}$.
20. $\dfrac{2x^2-18y^2}{x^2+6xy+9y^2}$.
21. $\dfrac{a^3-b^3}{2a-2b}$.
22. $\dfrac{ax+ay}{3x^3+3y^3}$.
23. $\dfrac{27x^3-8y^3}{9x^2-4y^2}$.

Reduce to lowest terms with as few minus signs as possible remaining in the numerator and denominator.

24. $\dfrac{-3}{a}$.
25. $\dfrac{-2x-2y}{a+b}$.
26. $\dfrac{2a+2d}{-3a-3d}$.
27. $\dfrac{-2x-2y}{(x+y)^2}$.
28. $\dfrac{ax-5a}{5-x}$.
29. $\dfrac{(d-c)^2}{c^2-cd}$.
30. $\dfrac{u^2-v^2}{3v-3u}$.
31. $\dfrac{x^2-6x+9}{18-2x^2}$.
32. $\dfrac{8x^3-y^3}{(y-2x)^2}$.
33. $\dfrac{2x^2+3x-9}{18-8x^2}$.
34. $\dfrac{9-15x+4x^2}{6c+6d-8cx-8dx}$.
35. $\dfrac{11x-15-2x^2}{x^3-27}$.
36. $\dfrac{15cx+20dx-9c-12d}{10x^2+29x-21}$.
37. $\dfrac{8x^3-2x+20x^2-5}{3d-12dx+12dx^2}$.
38. $\dfrac{8a^3+27b^3}{12ab+4a^2+9b^2}$.
39. $\dfrac{x^4-10b^2x^2+9b^4}{2bx^2-2b^3-x^3+b^2x}$.
40. $\dfrac{cx^4+cy^4+cx^2y^2}{x^6-y^6}$.

84. Lowest common multiple of polynomials

The LCM of two or more integral rational polynomials is defined as the polynomial of *lowest* degree in all the literal numbers, with *smallest* integral coefficients, which has each given polynomial as a factor. Two results for a LCM which differ only in sign will be considered essentially identical because usually the sign of a LCM is of no importance. To find a LCM, first factor the polynomials.

ILLUSTRATION 1. The LCM of

$$2(3 - x)(3 + x), \qquad 4(x - 3)(x - 1), \quad \text{and} \quad 3(x - 3)^2$$

is $4 \cdot 3(x - 3)^2(x + 3)(x - 1)$. We did not consider $(3 - x)$ and $(x - 3)$ as distinct factors because $3 - x = -(x - 3)$.

The LCD of two or more fractions is the LCM of their denominators. We shall deal with the notion of a LCM only where it is a LCD.

Note 1. The **highest common factor (HCF)** of two or more integral rational expressions is the expression of *highest* degree, with *largest* integral coefficients, which is a factor of each of the given expressions. Thus, the HCF of $6x^2y^3$ and $4xy^4$ is $2xy^3$. We shall not find it essential to use the HCF terminology.

85. Addition of fractions with polynomial denominators

SUMMARY. *To express a sum of fractions as a single fraction:*

1. *Find the* LCD; *that is, factor each denominator and form the product of all different prime factors, giving to each factor the highest exponent with which it appears in any denominator.*

2. *For each fraction, divide the* LCD *by the denominator and then multiply both numerator and denominator by the resulting quotient, to express the fraction as an equal one having the* LCD.

3. *Combine the new numerators just obtained, with each numerator placed in parentheses preceded by the sign of its fraction, and divide by the* LCD.

Note 1. To check the addition of fractions, substitute explicit values for the literal numbers in the given sum and the final result.

EXAMPLE 1. Express as a single fraction:

$$\frac{4x}{x^2 - 9} - \frac{3x}{x^2 + x - 6}. \tag{1}$$

Solution. 1. Factor the denominators:

$$x^2 - 9 = (x - 3)(x + 3); \qquad x^2 + x - 6 = (x + 3)(x - 2).$$

Hence, LCD $= (x - 3)(x + 3)(x - 2)$.

2. In the 1st fraction, $\quad$ LCD $\div (x^2 - 9) = x - 2$.

3. In the 2d fraction, $\quad$ LCD $\div (x^2 + x - 6) = x - 3$.

4. We multiply numerator and denominator by $x - 2$ in the 1st fraction, and by $x - 3$ in the 2d fraction:

$$\frac{4x}{x^2 - 9} - \frac{3x}{x^2 + x - 6}$$

$$= \frac{4x(x - 2)}{(x - 3)(x + 3)(x - 2)} - \frac{3x(x - 3)}{(x - 3)(x + 3)(x - 2)} \tag{2}$$

$$= \frac{4x(x - 2) - 3x(x - 3)}{(x - 3)(x + 3)(x - 2)} = \frac{x^2 + x}{(x - 3)(x + 3)(x - 2)}. \tag{3}$$

Check. When $x = 4$, we obtain:

In (1): $$\frac{16}{16 - 9} - \frac{12}{16 + 4 - 6} = \frac{16}{7} - \frac{6}{7} = \frac{10}{7}.$$

For the result in (3): $$\frac{16 + 4}{(4 - 3)(4 + 3)(4 - 2)} = \frac{10}{7}, \textit{ which checks.}$$

Comment. With practice, the student should be able to omit details such as those on the right in (2).

Illustration 1. In the following addition of fractions, we change signs in the second denominator in order to exhibit the identical nature of two factors in the denominators:

$$\frac{5}{9c - 6d} + \frac{7}{4d - 6c} = \frac{5}{3(3c - 2d)} - \frac{7}{6c - 4d} = \frac{5}{3(3c - 2d)} - \frac{7}{2(3c - 2d)}$$

$$= \frac{(5\cdot 2) - (7\cdot 3)}{3\cdot 2(3c - 2d)} = -\frac{11}{6(3c - 2d)}.$$

EXERCISE 39

Change the fraction to an equal one with the specified denominator.

1. $3x/(x - 2)$; new denominator, $(x + 3)(x - 2)$.

2. $2y/(y - 4)$; new denominator, $(y - 4)(3y - 1)$.

3. $3x/(2x - 3)$; new denominator, $4x^2 - 9$.

4. $2/(a + 2)$; new denominator, $2a^2 + a - 6$.

5. $(3 - a)/(2 - a)$; new denominator, $2a - 4$.

Combine into a single fraction in lowest terms. Where letters are involved, check by substitution when directed by the instructor.

6. $\frac{7}{10} - \frac{11}{30} + \frac{4}{5}.$

7. $\frac{5x}{2a} - \frac{3x}{4b}.$

8. $\frac{2}{3} - \frac{3x+2}{x-5}.$

9. $\frac{1}{3(a-b)} + \frac{2}{5(a-b)}.$

10. $\frac{3}{7x+7y} - \frac{2}{5x+5y}.$

11. $\frac{5}{3x-3y} - \frac{2}{5x-5y}.$

12. $\frac{a}{a-b} - \frac{b}{a+b}.$

13. $\frac{2x}{3x-2y} - \frac{3y}{3x+2y}.$

14. $\frac{2x-1}{2x+3} - \frac{x+2}{2-3x}.$

15. $\frac{3}{2c-6d} + \frac{2}{3d-c}.$

16. $\frac{3}{2a-4b} - \frac{5}{6b-3a}.$

17. $\frac{x}{4x^2-1} + \frac{4}{6x-3}.$

18. $\frac{4}{3x-2} - \frac{2}{2x+3}.$

19. $\frac{5}{3x} - \frac{2x-3}{6x+6}.$

20. $\frac{2a}{9a^2-d^2} - \frac{3}{6a-2d}.$

21. $\frac{3}{2x-1} + \frac{5}{3x+3}.$

22. $\frac{3}{2x-4y} + \frac{5}{x^2-4y^2}.$

23. $3a - \frac{3-7a}{2a-3} + 1.$

24. $1 - \frac{6}{y} + \frac{1}{y^2-4y}.$

25. $\frac{3x}{2x+2y} + \frac{4}{x^2-y^2}.$

26. $\frac{5}{4x-x^2} + \frac{10}{3x^2-48}.$

27. $\frac{2a-n}{2a-2n} + \frac{3a-4n}{6n-6a}.$

28. $\frac{a-4}{2a-4} + \frac{2-11a}{2-a}.$

29. $\frac{5x}{x+4} - \frac{4x^2+2x-1}{x^2+x-12}.$

30. $\frac{2x+1}{x^2+4x-60} - \frac{2}{x-6}.$

31. $\frac{1}{3n-3} - \frac{n+6}{n^2+3n-4}.$

32. $\frac{a-2}{a^2-16} - \frac{a+2}{a^2+8a+16}.$

33. $\frac{2c-3}{2c^2-18} + \frac{4}{3c^2-11c+6}.$

34. $\frac{x+5x^2}{x^3-y^3} + \frac{3}{2x-2y}.$

35. $\frac{2x}{8x^3-27} - \frac{5}{4x^2-12x+9}.$

36. $\frac{x^2}{x^3+8} - \frac{2x}{x^2-2x+4}.$

37. $\frac{3x^2}{x^4-4} + \frac{5x^2-3}{2x^4+x^2-6}.$

38. $\frac{2x^3-3}{2x^6+3x^3-2} - \frac{x^3+3}{x^6-4}.$

39. $\dfrac{3x-2}{2x^2-x-3}-\dfrac{2x+5}{3x^2+6x+3}+2.$

40. $\dfrac{x-2}{x^2+x-6}-\dfrac{3x+5}{2x^2-x-6}+\dfrac{2x-1}{4x^2-9}.$

41. $\dfrac{3x+7}{8x^2-18}+\dfrac{3-4x}{2x^2-9x+9}-\dfrac{1}{2x^2-3x-9}.$

42. $\dfrac{3a+2b}{6a^2-ab-b^2}+\dfrac{a+3b}{3a^2+7ab+2b^2}-\dfrac{a-2b}{2a^2+3ab-2b^2}.$

43. $\dfrac{r-a}{r^2-6ar+9a^2}-\dfrac{2r-a}{r^2-9a^2}+3a.$

86. Factoring in multiplication or division of fractions

To multiply or divide fractions involving polynomials, factor the numerators and denominators and divide out all common factors from the numerator and denominator of the final result.

ILLUSTRATION 1. $\dfrac{2x^2+7x-15}{2x^2-3x-14}\cdot\dfrac{2x^2-19x+42}{8x-12}$

$$=\frac{(2x-3)(x+5)}{(2x-7)(x+2)}\cdot\frac{(2x-7)(x-6)}{4(2x-3)}=\frac{(x+5)(x-6)}{4(x+2)},$$

where we divided both numerator and denominator by $(2x-3)(2x-7)$.

ILLUSTRATION 2. $\dfrac{\dfrac{xy^2-y^3}{x^3+x^2y}}{\dfrac{x^2-2xy+y^2}{x^2-xy-2y^2}}=\dfrac{xy^2-y^3}{x^3+x^2y}\cdot\dfrac{x^2-xy-2y^2}{x^2-2xy+y^2}$

$$=\frac{y^2(x-y)}{x^2(x+y)}\cdot\frac{(x-2y)(x+y)}{(x-y)^2}=\frac{y^2(x-2y)}{x^2(x-y)},$$

where we divided both numerator and denominator by $(x-y)(x+y)$.

ILLUSTRATION 3. $\dfrac{2x-4}{x^2-5}\div(x-2)=\dfrac{2x-4}{x^2-5}\div\dfrac{x-2}{1}$

$$=\frac{2(x-2)}{x^2-5}\cdot\frac{1}{x-2}=\frac{2}{x^2-5},$$

where we divided out $(x-2)$.

Whenever a mixed expression is involved in the numerator or denominator of a fraction, or as a factor in a product, it is advisable to change the mixed expression to a single fraction as the first step in simplification.

EXAMPLE 1. Reduce to a simple fraction: $\dfrac{u^2 - \frac{25}{9}}{u - \frac{5}{3}}$.

SOLUTION. Express the numerator and denominator of the complex fraction as simple fractions and divide:

$$\frac{u^2 - \dfrac{25}{9}}{u - \dfrac{5}{3}} = \frac{\dfrac{9u^2 - 25}{9}}{\dfrac{3u - 5}{3}} = \frac{9u^2 - 25}{9} \cdot \frac{3}{3u - 5}$$

$$= \frac{(3u - 5)(3u + 5)}{9} \cdot \frac{3}{(3u - 5)} = \frac{3u + 5}{3}.$$

Comment. A somewhat shorter solution is obtained if, as the first step, we multiply both numerator and denominator by the LCD, 9, of the fractions involved in them.

ILLUSTRATION 4. $$\left(1 + \frac{3x}{2}\right) \div \left(2 - \frac{9x^2}{2}\right) = \frac{2 + 3x}{2} \div \frac{4 - 9x^2}{2}$$

$$= \frac{2 + 3x}{2} \cdot \frac{2}{(2 + 3x)(2 - 3x)} = \frac{1}{2 - 3x}.$$

EXERCISE 40

Perform the indicated operation and reduce to a simple fraction in lowest terms. Check by substituting values for the letters, where directed by the instructor.

1. $\dfrac{3a - 3b}{2a + 4b} \cdot \dfrac{a + 2b}{a - b}$.

2. $\dfrac{2c - 4d}{b - 3} \cdot \dfrac{ab - 3a}{bc - 2bd}$.

3. $\dfrac{hx - hy}{ab - ac} \cdot \dfrac{cw - bw}{3x - 3y}$.

4. $(h^2 - x^2) \cdot \dfrac{5w}{ch - cx}$.

5. $\dfrac{x - 1}{x^2 - 4x} \cdot (x^2 - 16)$.

6. $\dfrac{3c - bc}{5w - aw} \div \dfrac{3a - ab}{5k - ak}$.

7. $\dfrac{9y^2 - 1}{y^2 - 16} \div \dfrac{6y - 2}{y^2 + 4y}$.

8. $\dfrac{2x - 2y}{6x + 3y} \div \dfrac{(x - y)^2}{4x^2 - y^2}$.

9. $(5x - 3x^2) \div \dfrac{25 - 9x^2}{x + 3}$.

10. $\dfrac{h^2 - 9}{3x - 3y} \div \dfrac{h^2 - 6h + 9}{y^2 - x^2}$.

11. $\dfrac{\dfrac{x^2 - 4y^2}{2x - 6}}{\dfrac{x^2 + 2xy}{x^2 - 9}}$.

12. $\dfrac{\dfrac{a^2 - b^2}{(a + b)^2}}{\dfrac{ac - bc}{a^2 + ab}}$.

13. $\dfrac{\dfrac{a^3 + b^3}{2a - 3b}}{\dfrac{2a + 2b}{4a^2 - 9b^2}}$.

14. $\dfrac{\dfrac{3x-1}{9x^2-1}}{4x+5}$.

15. $\dfrac{\dfrac{ax+bx}{b^2-a^2}}{3x}$.

16. $\dfrac{\frac{2}{5}-a}{\dfrac{4}{a^2}-25}$.

17. $\dfrac{\dfrac{a}{b}-2}{\dfrac{a}{b}+3}$.

18. $\dfrac{\dfrac{a^2}{b}-b}{\dfrac{a}{b}+1}$.

19. $\dfrac{\dfrac{a}{2b}-\dfrac{1}{3}}{\dfrac{6}{b}-\dfrac{4}{a}}$.

20. $\dfrac{c^2-\dfrac{9}{d^2}}{1-\dfrac{3}{cd}}$.

21. $\dfrac{\dfrac{2}{x}-\dfrac{3}{y}}{\dfrac{4}{x^2}-\dfrac{9}{y^2}}$.

22. $\dfrac{3x-\dfrac{2}{y}}{9y^2-\dfrac{4}{x^2}}$.

23. $\dfrac{2x+\dfrac{1}{5y}}{10x^2-\dfrac{1}{10y^2}}$.

24. $\dfrac{\dfrac{2a}{a+b}-1}{\dfrac{a}{a+b}-1}$.

25. $\dfrac{1-\dfrac{8}{u^3}}{1-\dfrac{2}{u}}$.

26. $\dfrac{\dfrac{a}{x}-\dfrac{x}{a}}{\dfrac{a^2}{x^2}-\dfrac{x}{a}}$.

27. $\dfrac{\dfrac{a}{2}-\dfrac{1}{2a}}{\dfrac{a^3}{4}-\dfrac{1}{4a}}$.

28. $\dfrac{\dfrac{2x^2+5x-12}{x^2+2x+1}}{\dfrac{2x^2+x-6}{2x^2+5x+3}}$.

29. $\left(1+\dfrac{4}{x-2}\right)\left(3-\dfrac{2}{x+2}\right)$.

30. $\left(y+\dfrac{2x}{3}\right)\div\left(\dfrac{9y}{x}-\dfrac{4x}{y}\right)$.

31. $\left(\dfrac{y^2}{x^2}-\dfrac{x}{y}\right)\div\left(\dfrac{1}{2y}-\dfrac{x}{2y^2}\right)$.

32. $\left(1-\dfrac{25d^2}{a^2}\right)\div\dfrac{ax+5dx}{b-9a}$.

33. $\dfrac{1-\dfrac{3}{2x}}{2x^2+5x-12}$.

34. $\dfrac{\dfrac{1}{2}-\dfrac{3y}{4x}}{4x^2-12xy+9y^2}$.

35. $\dfrac{nx+vx-an-av}{n^2-v^2}\div(cx-ac)$.

36. $\dfrac{ab-av}{5a-x}\div(b^2+3bv-4v^2)$.

37. $\left(y-1-\dfrac{6}{y}\right)\div\left(1+\dfrac{2}{y}-\dfrac{15}{y^2}\right)$.

38. $\left(\dfrac{2}{x}+\dfrac{5}{x^2}-\dfrac{12}{x^3}\right)\div\left(4-\dfrac{8}{x}+\dfrac{3}{x^2}\right)$.

39. $(2a+5b)\cdot\dfrac{6ab}{4a^2+20ab+25b^2}$.

40. $\dfrac{2x^2y}{15x^2+7xy-4y^2}\cdot(3x-y)$.

41. $\left(\dfrac{a^4 - 81b^4}{a^2c - 3abc + 9b^2c} \cdot \dfrac{a + 3b}{a^2 - 6ab + 9b^2}\right) \div \dfrac{a^2 + 6ab + 9b^2}{a^3 + 27b^3}$.

42. $\dfrac{c^4 - 2c^3d + 4c^2d^2}{ac + 6bd - 2ad - 3bc} \div \left(\dfrac{c^2d - c^3}{c^2 - 4d^2} \cdot \dfrac{c^3 + 8d^3}{a^3 - 27b^3}\right)$.

43. $\dfrac{\dfrac{6}{b^2} - \dfrac{1}{ab} - \dfrac{12}{a^2}}{3a + 4b}$.

44. $\dfrac{5c + \dfrac{8d^3}{25c^2}}{5ac + 2ad}$.

45. $\dfrac{z - \dfrac{16a^2}{z}}{z - 8a + \dfrac{16a^2}{z}}$.

46. $\dfrac{\dfrac{2x}{a} + 9 - \dfrac{5a}{x}}{\dfrac{6x^2}{a^2} + \dfrac{x}{a} - 2}$.

47. $\dfrac{16x^4 - 81y^4}{3a^2b} \div (2x^2 - 5xy + 3y^2)$.

48. $\dfrac{12x^4 + 7x^2 - 10}{3u^2v^3} \div (9x^4 - 4)$.

Find the reciprocal of the expression and reduce the result to a simple fraction in lowest terms.

49. $\left(1 - \dfrac{3x}{2x + 2}\right)$.

50. $\left(\dfrac{3x + 1}{2x - 2} - 5 + \dfrac{3}{x + 1}\right)$.

Reduce to a simple fraction in lowest terms.

51. $\dfrac{\dfrac{a^4}{b^2} - b^2}{\dfrac{3}{b^4} + \dfrac{1}{a^2b^2} - \dfrac{2}{a^4}}$.

52. $\dfrac{\dfrac{x^4 + 4x^2 + 8}{x^2 - 4} - 1}{\dfrac{x^2}{x + 2} + 1}$.

53. $\left(\dfrac{a - 2}{a} - \dfrac{2}{a + 3}\right)\left(\dfrac{2}{a + 2} - \dfrac{3}{3 - a}\right)$.

54. $\dfrac{1 + \dfrac{1}{x}}{x - \dfrac{1}{2x + \dfrac{x + 1}{x}}}$.

55. $\dfrac{3 - \dfrac{2a}{5a - 1}}{4a - \dfrac{2a}{1 - \dfrac{a}{1 + 2a}}}$.

56. $\dfrac{2a^2}{5a - \dfrac{4a - 1}{1 + \dfrac{2a + 5}{3a - 2}}}$.

57. $\dfrac{1 + \dfrac{2a^2}{5a - 3}}{2a + \dfrac{\dfrac{4a - 6a^2}{a - 1} - 3}{3 + \dfrac{1}{a - 1}}}$.

87. Equations involving fractions

To solve an equation involving fractions, we proceed as follows.

1. *Factor all denominators and form the* LCD *in factored form.*

2. *Enclose each numerator in parentheses and multiply both sides of the equation by the* LCD *to clear the equation of fractions.*

3. *Remove parentheses and solve.*

EXAMPLE 1. Solve:
$$\frac{2x}{2x+3} - \frac{2x}{2x-3} = \frac{2x+28}{4x^2-9}. \tag{1}$$

SOLUTION. 1. The LCD is $(2x + 3)(2x - 3)$, or $(4x^2 - 9)$.

2. Multiply both sides by the LCD:
$$2x(2x - 3) - 2x(2x + 3) = 2x + 28, \tag{2}$$
because
$$(2x + 3)(2x - 3)\frac{2x}{2x-3} = 2x(2x + 3);\ \text{etc.}$$

3. Expand in (2) and collect terms:
$$4x^2 - 6x - 4x^2 - 6x = 2x + 28; \quad -28 = 14x; \quad x = -2.$$
The student should check by substituting $x = -2$ in (1).

88. Operations leading to extraneous roots

A. *If both members of an equation are divided by an expression involving the unknowns, the new equation may have fewer roots than the original equation.*

ILLUSTRATION 1. By substitution, we verify that $x = 1$ and $x = 2$ are roots of $x^2 - 3x + 2 = 0$. On dividing both sides by $(x - 2)$ we obtain
$$\frac{x^2 - 3x + 2}{x - 2} = 0, \quad \text{or} \quad \frac{(x-2)(x-1)}{x-2} = 0, \quad \text{or} \quad x - 1 = 0.$$
The final equation has *just one* root, $x = 1$. The root $x = 2$ was lost by the division.

In solving algebraic equations, we usually avoid operations of Type A in order that roots may not be lost.*

B. *If both members of an equation are multiplied by an expression involving the unknowns, the new equation thus obtained may have more solutions than the original equation.*

* See Note 4 in the Appendix for a "proof" that $2 = 1$, in which the fallacy involves an operation of Type A which conceals a division by zero.

ILLUSTRATION 2. The equation $x - 3 = 0$ has *just one* root, $x = 3$. If both sides of $x - 3 = 0$ are multiplied by $(x + 2)$ we obtain

$$(x + 2)(x - 3) = 0, \quad \text{or} \quad x^2 - x - 6 = 0.$$

By substitution, we verify that this equation has *two* roots, $x = 3$ and $x = -2$. The root -2 was introduced by the multiplication.

A value of the unknown, such as $x = -2$ in Illustration 2, which satisfies a derived equation but does *not* satisfy the original equation, is called an **extraneous root.**

Whenever an operation of Type B is employed, *test all values obtained to reject extraneous roots, if any.*

EXAMPLE 1. Solve:
$$\frac{x}{x^2 - 1} - \frac{1}{x^2 - 1} + \frac{2}{x + 1} = 0.$$

SOLUTION. The LCD is $x^2 - 1$; multiply both sides by $x^2 - 1$:

$$x - 1 + 2(x - 1) = 0; \quad 3x = 3; \quad \text{or} \quad x = 1.$$

TEST. Since $x = 1$ makes $x^2 - 1 = 0$ in the denominators of the given equation, 1 *cannot be accepted as a root* because division by zero is not admissible. Hence, 1 is an extraneous root and therefore the given equation has *no root.*

EXAMPLE 2. On a river whose current flows at the rate of 3 miles per hour, a motorboat takes as long to travel 12 miles downstream as to travel 8 miles upstream. At what rate could the boat travel in still water?

SOLUTION. 1. Let x miles per hour be the rate of the boat in still water. Then the rate of the boat in miles per hour going upstream is $(x - 3)$ and downstream is $(x + 3)$.

2. From the standard equation $d = vt$ of uniform motion, we obtain $t = d/v$. Hence, the time in hours for traveling

8 miles upstream is $\dfrac{8}{x - 3}$;

12 miles downstream is $\dfrac{12}{x + 3}$.

3. Hence,
$$\frac{8}{x - 3} = \frac{12}{x + 3}. \tag{1}$$

4. Multiply both sides of (1) by $(x - 3)(x + 3)$:

$$8(x + 3) = 12(x - 3); \quad 8x + 24 = 12x - 36;$$
$$4x = 60; \quad x = 15. \tag{2}$$

Thus, the boat travels 15 miles per hour in still water.

EXERCISE 41

Each equation will reduce to a linear equation if cleared of fractions properly. This reduction may be prevented and extraneous roots may be introduced if unnecessary factors are employed in the LCD. *Solve each equation and check.*

1. $\dfrac{6}{x-2}=\dfrac{1}{2}.$

2. $\dfrac{x}{x-3}=\dfrac{3}{2}.$

3. $\dfrac{7}{x-2}=\dfrac{5}{x}.$

4. $\dfrac{7}{x}-\dfrac{3}{x}=4.$

5. $\dfrac{17}{2z}-\dfrac{1}{z}=30.$

6. $\dfrac{3}{4z}=\dfrac{2}{5z^2}.$

7. $2-\dfrac{5x+7}{2x-2}=0.$

8. $\dfrac{4t-5}{6t+1}=\dfrac{2t-3}{3t+4}.$

9. $\dfrac{1+8x}{2x-1}-\dfrac{20x}{5x-3}=0.$

10. $\dfrac{2}{x+2}=\dfrac{5}{2x+2}.$

11. $\dfrac{4}{3h+14}=\dfrac{1}{2h+1}.$

12. $\dfrac{x}{3x-3}=\dfrac{x-3}{3x-6}.$

13. $\dfrac{2}{x^2-x}=\dfrac{1}{x^2-1}.$

14. $\dfrac{t-3}{t-1}=\dfrac{t^2+20-9t}{t^2+t-2}.$

15. $\dfrac{2x}{x-1}=2+\dfrac{5}{2x}.$

16. $\dfrac{2x+1}{3x-1}=\dfrac{2x^2-x+14}{3x^2+5x-2}.$

17. $\dfrac{1}{z-1}=1-\dfrac{4z-6}{5z-5}.$

18. $\dfrac{1}{2x+1}=\dfrac{1}{4x^2-1}-\dfrac{1}{2x-1}.$

19. $\dfrac{2z^2-3z}{z^3-8}=\dfrac{2}{z-2}.$

20. $\dfrac{6x^2+6}{6x^2-x-2}=\dfrac{3-3x}{3x-2}+2.$

21. $\dfrac{x}{.3+x}-.6=\dfrac{.3-x}{.3+x}.$

22. $\dfrac{4x^2+3}{8x^3+1}=\dfrac{1}{2x+1}.$

23. $\dfrac{x+3}{x-1}-\dfrac{2x+3}{x-5}=\dfrac{3-x^2}{x^2-6x+5}.$

24. $\dfrac{2}{4-9x^2}-\dfrac{3}{2+3x}=\dfrac{3x}{4-9x^2}.$

25. $\dfrac{x+1}{x-2}-\dfrac{4x+3}{2x-3}=\dfrac{7-2x^2}{2x^2-7x+6}.$

26. $\dfrac{2w-7}{2w^2+3w-14}=\dfrac{2w+3}{2w+7}-\dfrac{w-1}{w-2}.$

27. $\dfrac{6}{w+2}-\dfrac{5}{w}=\dfrac{5-4w}{w^2+2w}.$

28. $\dfrac{3}{x+4}-\dfrac{2}{x-5}=\dfrac{2x-1}{20+x-x^2}.$

29. In a certain fraction, the denominator exceeds the numerator by 5. If the numerator and denominator are both increased by 3, the fraction equals $\frac{3}{4}$. Find the original fraction.

30. On six quizzes in mathematics, a student has obtained 70% as his average score. How many scores of 85% each must be obtained to bring his average score up to 80%?

31. In one hour, Jones can plow $\frac{1}{6}$ of a field. If Jones and Smith both work, they can plow the field in 2 hours and 24 minutes. How many hours would it take Smith alone to plow the field?

32. When the wind velocity is 40 miles per hour, it takes a certain airplane as long to travel 320 miles against the wind as 480 miles with it. How fast can the airplane travel in still air?

33. When the wind velocity is 20 miles per hour, it takes a certain airplane 90% as long to travel with the wind to any destination as it would to return to the starting place against the wind. How fast can the airplane travel in still air?

34. A fuel tank has one intake pipe which fills it in 8 hours. A second intake pipe is installed and it is found that, when both are in use, they fill the tank in $2\frac{2}{3}$ hours. How long would it take the second pipe alone to fill the tank?

35. On a river whose current flows at the rate of 3 miles per hour, a motorboat takes as long to travel 12 miles downstream as to travel 8 miles upstream. At what rate could the boat travel in still water?

36. Two rivers flow at the rates of 3 miles and 4 miles per hour, respectively. It takes a man as long to row 13 miles downstream on the slower river as to row 15 miles downstream on the faster river. At what rate can he row in still water?

89. Solution of literal equations involving factoring

In solving a linear equation in a single unknown x, when other literal numbers occur in the equation, it may be necessary to factor either in clearing of fractions or in simplifying the final result.

EXAMPLE 1. Solve for x: $\quad b(b + x) = a^2 - ax.$

SOLUTION. 1. Expand: $\quad b^2 + bx = a^2 - ax.$

2. Add ax; subtract b^2: $\quad ax + bx = a^2 - b^2.$

3. Factor: $\quad x(a + b) = (a - b)(a + b).$

4. Divide by $(a + b)$: $\quad x = a - b.$

Comment. To check, substitute $x = a - b$ in the original equation.

EXAMPLE 2. Solve for w: $$\frac{aw+b^2}{b^2-a^2}+\frac{w+b}{a-b}=\frac{w+a}{a+b}.$$

SOLUTION. The LCD is $(b-a)(b+a)$. We first rewrite the equation to change the sign in the denominator $(a-b)$ and then clear of fractions by multiplying by the LCD:

$$\frac{aw+b^2}{b^2-a^2}-\frac{w+b}{b-a}=\frac{w+a}{a+b}.$$

Multiply by (b^2-a^2):

$$aw+b^2-(w+b)(b+a)=(w+a)(b-a). \qquad (1)$$

The student should expand in (1) and solve, to obtain $w=-a$.

EXERCISE 42

Solve for x or y or z, whichever appears.

1. $cx-3a=2h.$
2. $7x-a=3ax-5.$
3. $3az-bz=9a^2-b^2.$
4. $3bx-9b^2=2ax-4a^2.$
5. $4az-a^2=4z-1.$
6. $mnx-a=anx-m.$
7. $x^2-3n^2=(3n-x)^2.$
8. $b(b-x)=a^2+ax.$
9. $abx-a^2=b^2-abx.$
10. $bx-bd=ad-ax.$
11. $b(bx-a)=a^2x+b^2.$
12. $hz-h^2=kz-k^2.$
13. $2bx+6a^2=3ax+4ab.$
14. $ax-ab-a^2=b(x-2b).$
15. $acx+adx+d^2=c^2-bcx-bdx.$
16. $\dfrac{x(a-4b)-b^2}{a^2-b^2}+\dfrac{x+b}{a-b}=\dfrac{x+a}{a+b}.$
17. $\dfrac{c}{2x-d}=\dfrac{d}{2x-c}.$
18. $\dfrac{x-2a}{x+2b}=\dfrac{x+2b}{x-2a}.$
19. $\dfrac{2x}{2x+h}=\dfrac{2hx-h}{4x^2-h^2}+\dfrac{2x}{2x-h}.$
20. $\dfrac{2ac-x}{2ac+x}-1=\dfrac{2b}{a-b}.$
21. $\dfrac{x-a}{2b^2+ab-a^2}=\dfrac{2}{b+a}-\dfrac{x+2a}{2b^2+4ab+2a^2}.$
22. Solve $C=\dfrac{Kab}{b-a}$ for b.
23. Solve $S=\dfrac{rl-a}{r-1}$ for r.
24. Solve $s=\dfrac{a-b}{c-b}$ for b.

EXERCISE 43

Review of Chapters 4, 5, and 6

Perform the indicated operation and collect terms.

1. $(3x - 5y)(3x + 5y)$.
2. $(4x^2 - 3yz)(4x^2 + 3yz)$.
3. $(2x + 3)^2$.
4. $(x - 2z)^2$.
5. $(y^2 - 3w)^2$.
6. $(2a + 5b)^2$.
7. $(a - 4)(a^2 + 4a + 16)$.
8. $(2x - 3z)(4x^2 + 6xz + 9z^2)$.

Factor.

9. $y^2 - 25z^2$.
10. $4z^2 - 9h^2k^2$.
11. $z^2 - 8yz + 16y^2$.
12. $M^4 - 81y^4z^4$.
13. $a^3 - 27b^3$.
14. $8u^3 + 27v^3$.
15. $9y^2 + 12yz^2 + 4z^4$.
16. $y^2 + y - 12$.
17. $z^2 + 4z - 21$.
18. $6x^2 + x - 15$.
19. $2 - 12x^2 + 5x$.
20. $4h^2 - 28hw + 49w^2$.
21. $5z^2 - 30wz + 45w^2$.
22. $ab + 2bc + 3ad + 6cd$.
23. $2u^3 + 4a^2 - 2a - 4$.
24. $(m + w)^2 + 4m + 4w + 4$.
25. $x^2 - a^2 - 6ab - 9b^2$.
26. $x^2 + 4x + 4 - 9a^2$.

Reduce to a simple fraction in lowest terms.

27. $\dfrac{\dfrac{3}{x} + \dfrac{5}{y}}{\dfrac{2}{x} - \dfrac{3}{y}}$.
28. $\dfrac{1 - \dfrac{2}{3a}}{a^2 - \dfrac{4}{9}}$.
29. $\dfrac{27 + \dfrac{b^3}{x^3}}{\dfrac{9x}{b} - 3 + \dfrac{b}{x}}$.
30. $\dfrac{3x - 1}{2x - 5} - \dfrac{2x + 3}{3x - 1}$.
31. $\dfrac{2a - b}{a^2 - b^2} + \dfrac{5b}{b - a}$.
32. $\dfrac{3y - 2}{y - 4} - \dfrac{2y - 5}{3y + 1}$.
33. $\dfrac{3c - b}{c^2 - b^2} + \dfrac{3b}{b - c}$.

Solve for x and check.

34. $\dfrac{1 + 3x}{3} - \dfrac{2 + x}{5 + 2x} - x = 0$.
35. $\dfrac{12}{3 + 2x} - \dfrac{4 + 2x}{1 + 2x} + 1 = 0$.
36. $\dfrac{6x}{4x - 5} = \dfrac{7}{6x + 1} + \dfrac{3}{2}$.
37. $\dfrac{1 + 4x}{3 + 4x} + \dfrac{2x^2 + x}{2x^2 + 2} = 2$.
38. $\dfrac{5 - 4x}{7 - 2x} - \dfrac{12x^2 - 48x - 6}{4x^2 - 16x + 7} + 1 = 0$.
39. $\dfrac{a - b}{x} = \dfrac{a}{x + a} - \dfrac{b}{x + b}$.
40. $\dfrac{3x^2 - 2x^3 + 9}{x^3 + 27} = -\dfrac{3 + 2x}{3 + x}$.

CHAPTER 7

RECTANGULAR COORDINATES AND GRAPHS

90. Rectangular coordinates

On each of the perpendicular axes OX and OY in Figure 4, we lay off a scale with O as the zero point on both scales. In the plane of OX and OY we shall measure vertical distances in terms of the unit on OY and horizontal distances in terms of the unit on OX. We agree that horizontal distances will be considered *positive* if measured to the *right* and *negative* if to the *left;* vertical distances will be considered *positive* if measured *upward* and *negative* if *downward.* Let P be any point in the plane.

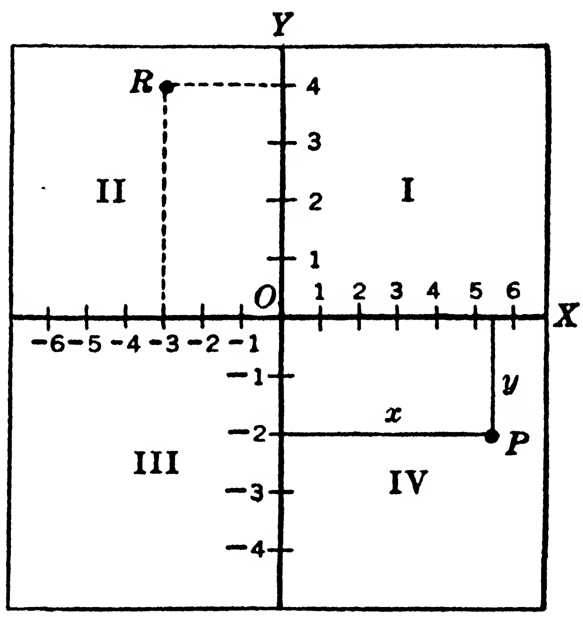

Fig. 4

The *horizontal coordinate,* or the **abscissa** of P, is the perpendicular distance, x, *from* OY *to* P; this directed distance is *positive* if P is to the *right* of OY and *negative* if P is to the *left* of OY.

The *vertical coordinate,* or the **ordinate** of P, is the perpendicular distance, y, *from* OX *to* P; this directed distance is *positive* if P is *above* OX and *negative* if P is *below* OX.

Each of the lines OX and OY is called a **coordinate axis,** and the abscissa and ordinate of P together are called the *rectangular coordinates of* P. The point O at which the axes intersect is called the **origin** of the coordinate system. When the axes are labeled OX and OY as in Figure 4, we sometimes refer to the abscissa as the x-coordinate and to the ordinate as the y-coordinate.

Notice that there is no necessity for using the same unit of length for the scales on OX and OY.

ILLUSTRATION 1. In Figure 4, the coordinates of P are $x = 5\frac{1}{2}$ and $y = -2$. The coordinates of a point are usually written together within parentheses with the abscissa first. Thus, we say that P is the point $(5\frac{1}{2}, -2)$. In Figure 4, R is the point $(-3, 4)$.

Note 1. The coordinate axes divide the plane into four parts called **quadrants,** which we number I, II, III, and IV, *counterclockwise.*

To *plot* a point, whose coordinates are given, means to locate the point and to mark it with a dot or a cross.

EXAMPLE 1. Plot the point $(-3, 4)$.

FIRST SOLUTION. At -3 on OX, erect a perpendicular to OX. Go up 4 vertical units on this perpendicular to reach the point R in quadrant II which is $(-3, 4)$.

SECOND SOLUTION. At $+4$ on OY, erect a perpendicular to OY. Go to the left 3 units on this perpendicular to reach $(-3, 4)$.

Note 2. The word *line* in this book will refer to a *straight* line unless otherwise specified.

EXERCISE 44

Plot the following points on a coordinate system on cross-section paper.

1. $(3, 4)$. **2.** $(3, 0)$. **3.** $(1, -2)$. **4.** $(-3, -5)$.

5. $(0, -2)$. **6.** $(-5, 0)$. **7.** $(0, 7)$. **8.** $(-3, 4)$.

9. $(-2, -3)$. **10.** $(-3, 5)$. **11.** $(4, -4)$. **12.** $(-2, 1)$.

13. Three corners of a rectangle are $(3, 4)$, $(-5, 4)$, and $(3, -1)$. Find the coordinates of the 4th corner and the area of the rectangle.

Find the area of a triangle with the given vertices.

14. $(4, 3)$; $(4, 7)$; $(-2, 3)$. **15.** $(0, -4)$; $(3, -4)$; $(3, 2)$.

16. $(-2, 1)$; $(3, 1)$; $(5, 5)$. **17.** $(0, 0)$; $(5, 3)$; $(5, 7)$.

18. A square, with its sides parallel to the coordinate axes, has one corner at $(-3, 2)$ and lies above and to the left of $(-3, 2)$. If the units of length on the axes are the same and if each side of the square is 4 units long, find the coordinates of the other corners.

In which quadrant does a point lie under the specified condition?

19. Both coordinates are negative.

20. The abscissa is negative and the ordinate is positive.

21. The abscissa is positive and the ordinate is negative.

22. A line is parallel to OX and passes through the point where $y = -3$ on OY. What is true about the ordinates of points on the given line?

23. A line is perpendicular to OX at the point where $x = 2$. What can be stated about the abscissas of points on the given line?

24. How far apart are the lines on which the abscissas of all points are -3 and 4, respectively?

25. How far apart are the lines on which the ordinates of all points are -7 and -3, respectively?

91. The function concept

We recall that, in a given problem, a constant is a number symbol whose value is not subject to change during the course of the discussion, and a variable is a number symbol which may take on different values. When desirable, we may think of a constant as a variable which can assume only one value.

If a first variable, x, and a second variable, y, are so related that, whenever a value is assigned to x, a corresponding value (or corresponding values) of y can be determined, we say that y is a **function** of x. Then x is called the *independent variable* and the second variable, y, which is a function of x, is called the *dependent variable.* To say that y is a function of x means that the value of y depends on the value of x.

ILLUSTRATION 1. In the formula $A = \pi r^2$ for the area of a circle, if r is a variable then A is a variable and A is a function of r.

Any formula in a variable x represents a function of x; the values of the function can be computed from its formula.

ILLUSTRATION 2. $(3x^2 + 7x + 5)$ is a function of x. If $x = 2$, the value of the function is $(12 + 14 + 5)$ or 31.

Note 1. If just *one* value of y corresponds to each value of x, we say that y is a *single-valued* function of x; if just *two* values of y correspond to each value of x, then y is a *two-valued* function of x; etc.

92. Graph of a function

Let y represent any function of x. Then, each pair of corresponding values of x and y can be taken as the coordinates of a point in an (x, y) coordinate system. This leads us to adopt the following terminology.

DEFINITION I. *The graph of a function, y, of x is the set of all points (or the* **locus** *of points) whose coordinates form pairs of corresponding values of x and y.*

To graph a function will mean to draw its graph. In graphing a function, we usually plot the values of the independent variable on the horizontal axis of the coordinate system.

A **linear function** of x is a polynomial of the *first* degree in x and hence has the form $ax + b$, where a and b are constants. In Illustration 1 below we meet a special case of the fact that **the graph of a linear function of x is a straight line.** This fact, whose proof we shall omit, is the basis for the name *linear* function of x.

ILLUSTRATION 1. If x is the independent variable, in order to graph the function $(\frac{3}{5}x - 3)$, we introduce y to represent it. That is, we let $y = \frac{3}{5}x - 3$. If $x = -5$, then $y = \frac{3}{5}(-5) - 3 = -6$. Hence, one point on the graph is $(-5, -6)$. Similarly, we let $x = 0$, -2, etc., and compute the corresponding values of y given in the following table. We plot $(-5, -6)$, $(-2, -4\frac{1}{5})$, etc., in Figure 5 and join them by a straight line, which is the graph of the function. From the graph, we read that the value of the function is *zero* (the graph crosses the x-axis) when $x = 5$. The function equals -2 when $x = 1\frac{2}{3}$, approximately.

$x =$	-5	-2	0	3	6
$y =$	-6	$-4\frac{1}{5}$	-3	$-1\frac{1}{5}$	$\frac{3}{5}$

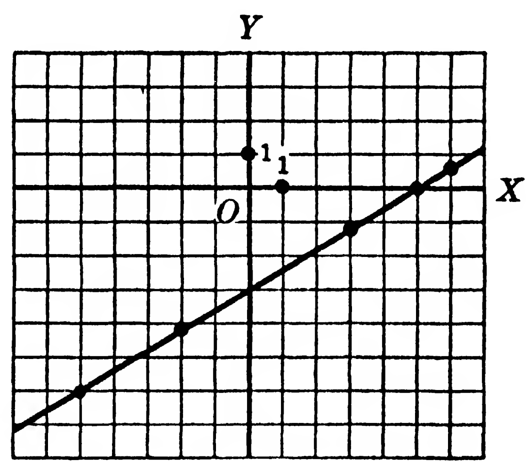

Fig. 5

If y is a linear function of x, we need only *two* pairs of values of x and y to obtain the graph, because a straight line is definitely located if we know two points on it. However, in graphing any linear function, we shall compute *three* values of the function in order to check the arithmetic involved. If the corresponding three points do not lie on a line, an error is indicated.

Note 1. In graphing, do not choose the position of the origin or the scales on the coordinate axes until after a reasonably complete table of values has been prepared. Then, make the appropriate selections of origin and scales so that as large a graph as possible may be placed on the available paper.

If a function of x is defined by a formula, in general its graph is a *smooth curve.** To graph such a function, we introduce some letter, such as y, to represent the function, compute a table of corresponding values of x and y, and draw a smooth curve through the corresponding points on a coordinate system.

ILLUSTRATION 2. To graph $x^2 - 4x + 6$, we let y represent the function,

$$y = x^2 - 4x + 6,$$

compute the following table of values, and plot the points. The graph, in Figure 6, is a curve called a *parabola.*

$x =$	-1	1	2	3	5
$y =$	11	3	2	3	11

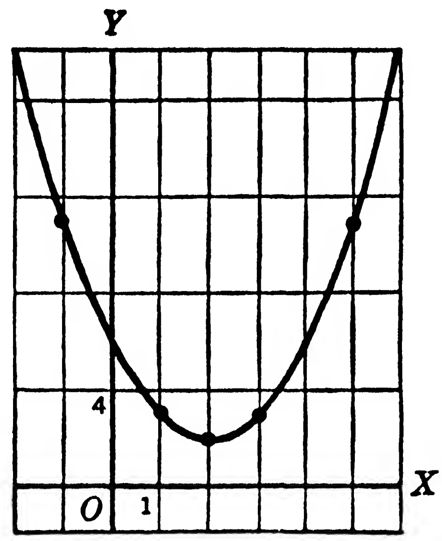

Fig. 6

93. Functions not defined by formulas

Functions not defined by formulas arise frequently. Sometimes the only information concerning a function consists of a table of corresponding values of the function and the independent variable, where the table may be obtainable by experimental means or observation. In drawing the graph of such a function, sketch a smooth curve through the points obtained from the given values, unless otherwise directed. Instead of drawing a smooth curve through the points, it is sometimes desirable to connect them by segments of straight lines and thus to obtain a *broken-line graph.*

Note 1. The intersection of the coordinate axes may be selected to represent any convenient value, *not* necessarily zero, on either scale.

ILLUSTRATION 1. The second row of the following table gives the general wholesale price index number of the United States Department of Labor for the critical depression months from June, 1930, to June, 1931. A value like 86.8 means 86.8% of the average level in 1926. To graph *the index*

* Or, in some cases, two or more *disconnected* smooth curves.

number as a function of the time, in Figure 7, we choose coordinate axes, with time plotted *horizontally* and index number *vertically*. We take 1 month to be the unit of time. We let the intersection (origin) of the axes represent June, 1930, on the axis of abscissas and 60 on the vertical axis, and assign units on the axes to suit the size of the figure. Then, for December, 1930, we plot the point (6, 78.4), etc. We join the plotted points by a reasonably smooth curve, which is the graph of the function. From the graph, extended as a guess to July, 1930, we estimate that the index number then was 68.6.

June '30	July	Aug.	Sept.	Oct.	Nov.	Dec.	Jan. '31	Feb.	Mar.	Apr.	May	June
86.8	84.0	84.0	84.2	82.6	80.4	78.4	77.0	75.5	74.5	73.3	71.3	70.0

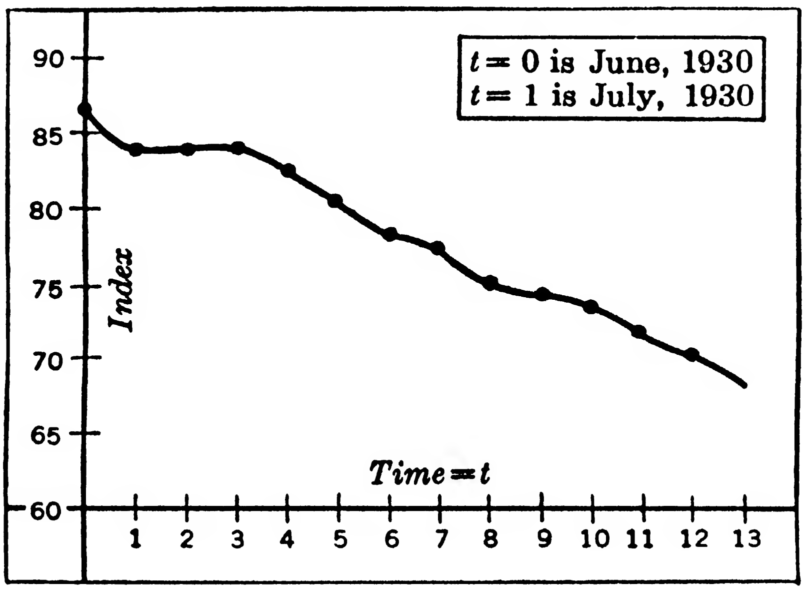

Fig. 7

EXERCISE 45

The letter x represents the independent variable in all problems where it appears. Clearly indicate the scale on each coordinate axis employed.

1. Graph the function $(2x + 3)$. From the graph, (a) read the values of the function when $x = 2\frac{1}{2}$ and $x = -3\frac{1}{2}$; (b) read the values of x corresponding to which the values of the function are 2, 0, and 3.

Graph the function of x and, from the graph, read the value of x for which the function equals zero.

2. $3x + 5$. **3.** $3 - 4x$. **4.** $-2 - 5x$. **5.** $3x$.

6. $-2x$. **7.** $-2 + 3x$. **8.** $4 - 2x$. **9.** $-3 - 2x$.

10. 7. **11.** -4. **12.** 0. **13.** x.

Hint for Problem 10. Any constant can be considered as a function of any variable x, with just one value for the function. The graph is horizontal.

14. Graph the function of y defined by $(3y - 4)$, with the y-axis horizontal and with z used as a label for the function.

15. Graph $(x^2 - 6x + 7)$ by computing its values for the following values of x: $-1, 0, 2, 3, 4, 6$, and 7. From the graph, (*a*) read the values of the function when $x = 5$ and $x = 1$; (*b*) read the values of x for which the function equals 0 or 10.

16. Graph $(-x^2 - 4x + 6)$ by computing its values for the following values of x: $-6, -5, -4, -3, -2, -1, 0, 1$, and 2. From the graph, read the values of x for which the function (*a*) equals 0; (*b*) equals -3.

17. The table gives the total mileage of hard-surfaced roads forming parts of state highway systems in the United States at the ends of various years. Graph the mileage as a function of the time.

Year	1926	1929	1931	1934	1939	1940	1942
Mileage	54,000	75,000	96,000	110,000	120,000	122,000	130,000

18. The table gives the time it takes money to double itself if invested at certain rates of interest, compounded semiannually. Graph the time as a function of the rate. From the graph, find the time for money to double at $3\frac{1}{2}\%$.

Time, years	$69\frac{1}{2}$	$46\frac{1}{3}$	$34\frac{3}{4}$	28	$23\frac{1}{4}$	$17\frac{1}{2}$	14	$11\frac{3}{4}$
Rate	1%	$1\frac{1}{2}\%$	2%	$2\frac{1}{2}\%$	3%	4%	5%	6%

19. The velocity of sound in air depends on the temperature of the air. By use of the following data, graph the velocity as a function of the temperature. From the graph, read the velocity if the temperature is 35°; 8.5°; 120°.

Velocity, ft. per sec.	1030	1040	1060	1080	1110	1140	1170
Temp. (Fahrenheit)	− 30°	− 20°	0°	20°	50°	80°	110°

Hint. Let the origin represent 1000 feet on the vertical axis.

20. The table gives the number of divorces per 1000 marriages in various years in continental United States. Graph the number of divorces as a function of the time.

Year	1890	1900	1916	1922	1930	1934	1937	1940
Divorces	62	81	108	131	170	157	173	169

21. The weight of a cubic foot of dry air at an atmospheric pressure of 29.92 inches of mercury, under various temperatures, is given in the following table, where weight is in pounds, and temperature is in degrees Fahrenheit. Graph the weight of air as a function of the temperature.

TEMP.	0°	12°	32°	52°	82°	112°	152°	192°	212°
WEIGHT	.0864	.0842	.0807	.0776	.0733	.0694	.0646	.0609	.0591

22. The following table gives the "*thinking distance*" t, and the "*braking distance*" b involved when a motorist, traveling at s miles per hour, decides to stop his car. The value of t is the distance traveled by the car in $\frac{3}{4}$ second, the interval which elapses between the instant an average driver sees danger and the instant he applies his brakes. The sum $d = t + b$ is the total distance the car will travel before stopping after danger is seen. On one coordinate system, draw graphs of t as a function of s and d as a function of s.

s (*mph*)	20	30	40	50	60	70
t (*feet*)	22	33	44	55	66	77
b (*feet*)	21	46	82	128	185	251

94. Functional notation

Sometimes we represent functions by symbols like $f(x)$, $H(x)$, $K(s)$, etc. The letter in parentheses tells what the independent variable is. The letter to the left is merely a convenient *name* for the function.

ILLUSTRATION 1. We read "$f(x)$" as "the f-function of x," or for short "f of x." We may represent $3x^2 - 5$ by $f(x)$ and write $f(x) = 3x^2 - 5$; we read this "*f of x is* $3x^2 - 5$." $H(y)$ would represent a function of y. For instance, we may let $H(y) = 7y^3 + 6$.

If $F(x)$ is any function of x and a is any value of x, then

$F(a)$ represents the value of $F(x)$ when $x = a$.

ILLUSTRATION 2. "$F(a)$" is read "F of a." If $F(x) = 3x^2 - 5 - x$,

$$F(3) = 3\cdot 3^2 - 5 - 3 = 19;$$
$$F(-3) = 3(-3)^2 - 5 + 3 = 25;$$
$$F(-b^2) = 3(-b^2)^2 - 5 - (-b^2) = 3b^4 - 5 + b^2;$$
$$[F(-2)]^2 = (12 - 5 + 2)^2 = 81;$$
$$5F(2) = 5(12 - 5 - 2) = 25.$$

A variable z is said to be a function of *two* variables x and y in case a value of z can be determined corresponding to each pair of values of x and y. Similarly, we may speak of a function of three variables, or of any number of variables. The functional notation just introduced for functions of a single variable is extended to functions of more than one variable.

ILLUSTRATION 3. $F(x, y)$ would be read "F of x and y" and would represent a function of the independent variables x and y. Thus, we may let

$$F(x, y) = x + 3y^2 + 2.$$

Then, $F(2, 1) = 2 + 3 + 2 = 7.$

EXERCISE 46

If $f(x) = 2x + 3$, find the value of each symbol.

1. $f(2)$. **2.** $f(-3)$. **3.** $f(-2)$. **4.** $f(\frac{1}{2})$. **5.** $f(-\frac{1}{4})$. **6.** $f(-\frac{3}{5})$.

If $G(z) = 2z - 3z^2$, find the value of the symbol or an expression for it.

7. $G(-3)$. **8.** $G(5)$. **9.** $G(\frac{1}{2})$. **10.** $G(a)$. **11.** $G(2c)$. **12.** $G(3x)$.

13. If $F(x) = x^2 - x + 3$, find $F(-2)$; $F(b)$; $F(c^2)$; $F(x - 2)$.

14. If $G(w) = \dfrac{3w + 2}{w}$, find $G(2)$; $3G(1)$; $[G(3)]^2$; $\dfrac{G(3)}{G(6)}$.

15. If $K(z) = \dfrac{z + 2}{z - 1}$, find $K(2)$; $2K(4)$; $[K(3)]^3$; $K\left(\dfrac{x}{y}\right)$.

16. If $F(x, y) = 3x + 2y$, find $F(2, -3)$; $F(-1, 4)$; $F(a, b)$.

17. If $F(x, y) = x^2 + 3xy$, find $F(-1, 2)$; $F(-3, -2)$; $F(c, 2b)$.

18. If $F(x) = x^2 - 3x$, find $F\left(\dfrac{a}{b}\right)$: $\dfrac{F(a)}{F(b)}$.

19. If $f(x) = x^2 - 4x + 5$, graph $f(x)$ by use of $f(-1)$, $f(0)$, $f(1)$, $f(2)$, $f(3)$, $f(4)$, and $f(5)$.

20. If $f(x) = x^3 - 12x + 3$, graph $f(x)$ by use of $f(-4)$, $f(-3)$, $f(-2)$, $f(-1)$, $f(0)$, $f(1)$, $f(2)$, $f(3)$, and $f(4)$.

95. Functions defined by equations

A **solution** of an equation in two variables x and y is a **pair of corresponding values** of x and y which satisfy the equation. Usually, an equation in two variables has *infinitely many* solutions.

ILLUSTRATION 1. Consider $3x - 5y = 15$. If $x = 3$, then $9 - 5y = 15$, or $y = -\frac{6}{5}$. Hence, $(x = 3, y = -\frac{6}{5})$ is a solution of the given equation. If $y = 0$, then $3x = 15$ or $x = 5$; hence $(x = 5, y = 0)$ is another solution. Thus, by substituting values for either variable and computing values of the other variable, we could find as many solutions as we might desire.

In case x and y are related by an equation, then usually we may think of y as a function of x and, likewise, of x as a function of y. This is true because, in general, for each value of either variable we can find corresponding values of the other variable by use of the equation. In particular, a *linear* equation in x and y defines either variable as a *linear* function of the other variable.

ILLUSTRATION 2. From $3x - 5y = 15$, on solving for x we obtain

$$x = 5 + \tfrac{5}{3}y;$$

on solving for y we obtain

$$y = \tfrac{3}{5}x - 3.$$

Hence x is a linear function of y and, equally well, y is a linear function of x.

96. Graphical representation of an equation

The **graph,** or the **locus,** of an equation in two variables x and y is the locus of all points whose coordinates (x, y) form solutions of the equation. If we think of x as an independent variable, the graph of the equation is *identical with the graph of the function, y, of x, defined by the equation.* In particular, if a, b, and c are constants, **the graph of the linear equation $ax + by = c$ is a straight line.** For, the graph of this equation is the graph of the linear function of x, or of the linear function of y, defined by the equation.

ILLUSTRATION 1. From $3x - 5y = 15$, we obtain $y = \frac{3}{5}x - 3$. The graph of $3x - 5y = 15$ is the graph of the linear function $\frac{3}{5}x - 3$; this graph is found in Figure 5, page 122.

The abscissa of any point where a graph on an (x, y) coordinate system meets the x-axis is called an **x-intercept** of the graph. The ordinate of any point where the graph meets the y-axis is called a **y-intercept** of the graph. To find the *x-intercept* (or *intercepts*) of the graph of an equation in x and y, place $y = 0$ in the equation and solve for x; to find the *y-intercept* (or *intercepts*), place $x = 0$ and solve for y.

SUMMARY. *To graph a linear equation in x and y:*

1. *Place $x = 0$ and compute y, to find the y-intercept.*

2. *Place $y = 0$ and compute x, to find the x-intercept.*

3. *Find any other solution of the equation and draw the line through the points determined on plotting the solutions obtained.*

ILLUSTRATION 2. To graph $3x - 5y = 15$, first let $x = 0$ and obtain $0 - 5y = 15$, or $y = -3$; hence, $(0, -3)$ is a point on the graph. If $y = 0$ then $3x - 0 = 15$, or $x = 5$; the x-intercept is 5, or $(5, 0)$ is a point on the graph. The graph is shown in Figure 5, page 122.

ILLUSTRATION 3. The graph of the equation $x - 5 = 0$ consists of all points (x, y) in the coordinate plane for which $x = 5$, and the value of y is of no importance because it does not occur in the equation. Hence, the graph of $x - 5 = 0$ is the line perpendicular to the x-axis at the point where $x = 5$, as shown in Figure 8.

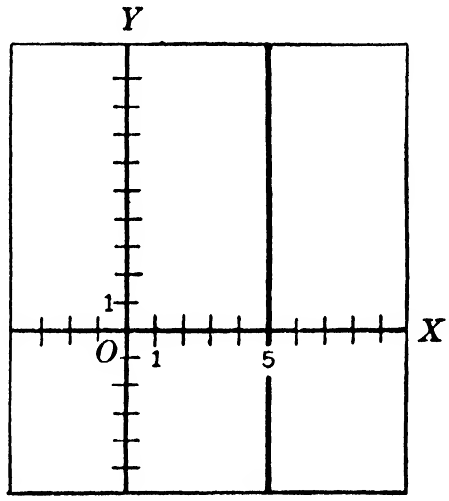

Fig. 8

97. Equation of a line

An *equation of a curve* on an (x, y) coordinate plane is an equation in the variables x and y *whose graph is the given curve.* If two equations have the same graph, in general the equations differ only in nonessential features. Hence, although a given curve may have infinitely many different equations, we shall refer to any one of these as *the* equation of the curve.

ILLUSTRATION 1. $3x + 2y = 7$ is the equation of a certain straight line. This line also is the graph of $6x + 4y = 14$ because these two equations have the same solutions.

Frequently we refer to a function of a variable x, or to an equation in x and y, by giving the function or equation the name of its graph.

ILLUSTRATION 2. Thus, we may refer to the *line* $3x + 2y = 7$, or to the *parabola* $y = x^2 - 4x + 6$ (see Figure 6, page 123).

We shall assume without proof the fact that the equation of any straight line on an (x, y) coordinate plane is of the form $ax + by = c$ where a, b, and c are constants. The equation of a line is a linear

relation between x and y which is true when and only when the point (x, y) is on the line.

ILLUSTRATION 3. The equation of the vertical line 3 units to the left of the y-axis is $x = -3$.

ILLUSTRATION 4. Let P, with coordinates (x, y), be any point on the line through $(0, 0)$ and $(1, 2)$. Then, from similar right triangles in Figure 9,

$$\frac{y}{x} = \frac{2}{1} \quad \text{or} \quad y = 2x;$$

this is the equation of the line.

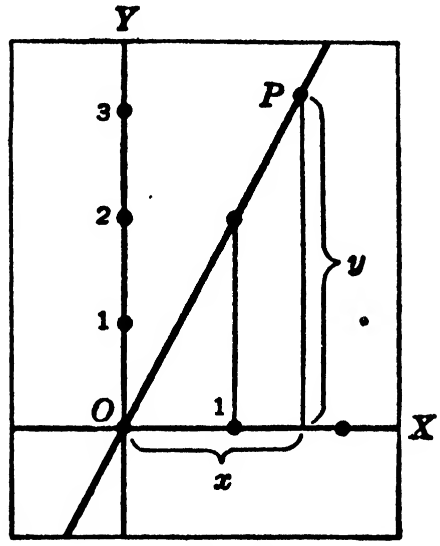

Fig. 9

EXERCISE 47

Graph each equation.

1. $3x + 2y = 6$.
2. $3y - 4x - 12 = 0$.
3. $3x - 5y = 0$.
4. $2x + 7y = 0$.
5. $3x - y = 9$.
6. $3x = 15 + 5y$.
7. $4x - 5y = 20$.
8. $2x = 3y$.
9. $5y + x = 10$.
10. $x = 7$.
11. $y = 5$.
12. $x = -3$.
13. $y = -4$.
14. $y = 0$.
15. $2x = y$.
16. $5x + 9 = 0$.
17. $3y + 4 = 0$.

Give the equation of the line satisfying the given condition.

18. The horizontal line (*a*) 6 units above OX; (*b*) 4 units below OX.

19. The vertical line (*a*) 5 units to the right of OY; (*b*) 4 units to the left of OY.

20. The line on which the ordinate of each point is (*a*) the same as its abscissa; (*b*) the negative of its abscissa.

Without graphing, find the coordinates of the points where the graph of the equation cuts the axes.

21. $3x + 5y = 15$.
22. $2x - 5y = 10$.
23. $-3x + 2y - 5 = 0$.

24. Find an expression for the linear function of y defined by the equation $2x + 7y = 9$.

25. Find an expression for the linear function of x defined by the equation $3x - 5y = 11$.

CHAPTER 8

SYSTEMS OF LINEAR EQUATIONS

98. Graphical solution of a system of two equations

A *solution of a system* of two equations in two unknowns, x and y, is a *pair* of corresponding values of x and y which satisfy both equations. If a system has a solution, the equations are called **simultaneous.**

EXAMPLE 1. Solve graphically:
$$\begin{cases} x - y = 5, & (1) \\ x + 2y = 2. & (2) \end{cases}$$

SOLUTION. 1. In Figure 10, AB is the graph of (1) and CD is the graph of (2). AB consists of all points whose coordinates satisfy (1) and CD consists of all points whose coordinates satisfy (2). Hence, the point of intersection, E, of AB and CD is the only point whose coordinates satisfy both equations.

2. We observe that E has the coordinates $(4, -1)$. Hence, $(x = 4, y = -1)$ is the only solution of the system. These values check when substituted in (1) and (2).

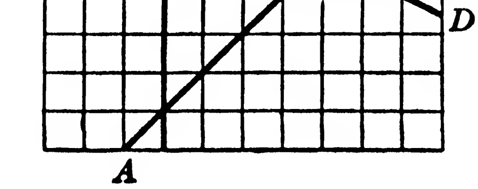

Fig. 10

SUMMARY. *To solve a system of two equations in two unknowns graphically:*

1. *Draw the graphs of the equations on one coordinate system.*

2. *Measure the coordinates of any point of intersection of the graphs; these coordinates form a solution of the system.*

Usually a system of two linear equations in two unknowns has just one solution, as was the case in Example 1, but the following special cases may occur.

A. *If the graphs of the equations are parallel lines, the system has no solution and the equations are called* **inconsistent** *equations.*

B. *If the graphs of the equations are the same line, each solution of either equation is also a solution of the other and hence the system has infinitely many solutions. In this case the equations are said to be* **dependent** *equations.*

Note 1. Usually a graphical solution gives only approximate results, because in obtaining them we estimate certain coordinates visually.

EXERCISE 48

Solve graphically. If there is no solution, or if there are infinitely many, state this fact with the appropriate reason.

1. $\begin{cases} x - y = 1, \\ y + 2x = -3. \end{cases}$ **2.** $\begin{cases} y + x = 2, \\ 2y - x = -5. \end{cases}$ **3.** $\begin{cases} y - 2x = 1, \\ 3y + 4x = 23. \end{cases}$

4. $\begin{cases} 2y - 3x = 0, \\ 4y + 3x = -18. \end{cases}$ **5.** $\begin{cases} 3x + 8 = 0, \\ 6x + 7y = 5. \end{cases}$ **6.** $\begin{cases} 5y - 3 = 0, \\ 10y + 3x = 4. \end{cases}$

7. $\begin{cases} 2y - 5x = 10, \\ 2y - 2x = 3. \end{cases}$ **8.** $\begin{cases} 2x - 3y = 0, \\ 5x + 7y = 0. \end{cases}$ **9.** $\begin{cases} 3x + 5y = 2, \\ 2x - 3y = 5. \end{cases}$

10. $\begin{cases} x + 2y = 4, \\ 3x - y = 6. \end{cases}$ **11.** $\begin{cases} 2x - y = 3, \\ 2y - 4x = 5. \end{cases}$ **12.** $\begin{cases} 3 = 2x - 3y, \\ 4x - 6 = 6y. \end{cases}$

13. $\begin{cases} x + y = 1, \\ 2x + 2y = 7. \end{cases}$ **14.** $\begin{cases} 3x - 4y = 5, \\ 6x - 8y = 3. \end{cases}$ **15.** $\begin{cases} x - 5y = 2, \\ 10y - 2x + 4 = 0. \end{cases}$

16. (*a*) Graph $x + 3y = 5$. (*b*) Multiply both sides of the equation by 2 and graph the new equation. (*c*) By inference, state how two linear equations are related if they have the same graph.

99. Elimination by addition or subtraction

Example 1. Solve for x and y:
$$\begin{cases} 4x + 5y = 6, & (1) \\ 2x + 3y = 4. & (2) \end{cases}$$

Solution. 1. Multiply (1) by 3: $12x + 15y = 18.$ (3)

2. Multiply (2) by 5: $10x + 15y = 20.$ (4)

3. Subtract, (3) − (4): $2x = -2;\quad x = -1.$ (5)

In obtaining equation 5, we have eliminated y by *subtraction.*

4. On substituting $x = -1$ in (2) we obtain $3y = 4 + 2$ or $y = 2.$

5. The solution of the system is $(x = -1, y = 2)$. The student should check this solution by substitution in (1) and (2).

Summary. *To solve a system of two linear equations by elimination by addition or subtraction:*

1. *In each equation, multiply both members, if necessary, by a properly chosen number to obtain two equations in which the coefficients of one unknown have the same absolute value.*

2. *Add, or subtract, corresponding sides of the two equations obtained in Step 1 so as to eliminate one unknown.*

3. *Solve the equation found in Step 2 for the unknown in it, and substitute the result in one of the given equations to find the other unknown.*

If two linear equations in x and y are *inconsistent* or *dependent*, then, in eliminating one unknown, the other will also be eliminated. If the equations are dependent, an identity $0 = 0$ results from this elimination. If the equations are inconsistent, a contradictory equation such as $0 = 36$ is obtained. We shall omit proving these facts but shall exhibit special cases of them.

Note 1. Hereafter, *to solve* a system of equations will mean to solve *algebraically*, unless otherwise stated.

If the given equations involve fractions, clear of fractions before applying the preceding method.

Example 2. Solve for x and y:

$$\begin{cases} 3x + 2y = 6, & (6) \\ 6x + 4y = 24. & (7) \end{cases}$$

Solution. 1. Multiply (6) by 2:

$$6x + 4y = 12. \tag{8}$$

2. Subtract, (7) − (8):

$$0 = 12. \tag{9}$$

Hence, the given equations are inconsistent because a contradictory statement, $12 = 0$, results from the assumption that a pair of values of x and y exists which satisfies (6) and (7).

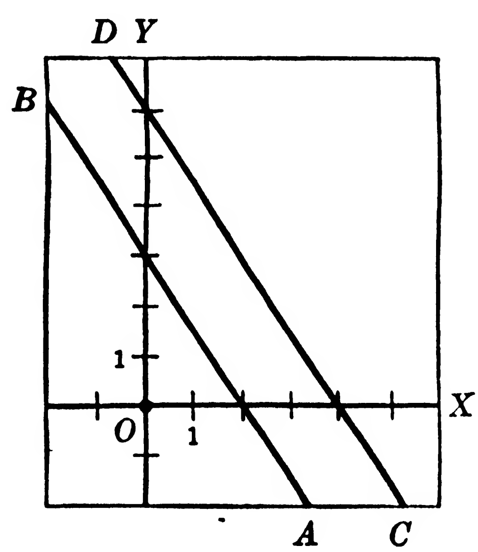

Fig. 11

Comment. In Figure 11, AB is the graph of (6) and CD is the graph of (7). It is observed that these lines are parallel and hence do not intersect, which agrees with the preceding algebraic proof that (6) and (7) have no solution.

EXERCISE 49

Solve by elimination by addition or subtraction and check.

1. $\begin{cases} 3x - y = 7, \\ 2x + 3y = 12. \end{cases}$
2. $\begin{cases} 2x = y - 6, \\ x + 2y = 2. \end{cases}$
3. $\begin{cases} 3x - 4y = 9, \\ 2x - 3y = 7. \end{cases}$
4. $\begin{cases} 5y - 2x = 0, \\ 3x + 2y = 0. \end{cases}$
5. $\begin{cases} 2x = 3y + 12, \\ 5x + 2y + 8 = 0. \end{cases}$
6. $\begin{cases} 6y = 3x + 10, \\ 9y = 6x + 14. \end{cases}$
7. $\begin{cases} 3x - 2y = 2, \\ 4y - 3x = 2. \end{cases}$
8. $\begin{cases} 2x = 3y + 9, \\ 2x + \frac{3}{2}y = \frac{15}{2}. \end{cases}$
9. $\begin{cases} \frac{4}{3}y + 2x = -3, \\ 3y + 2x = -\frac{1}{2}. \end{cases}$
10. $\begin{cases} \dfrac{x}{3} - \dfrac{y}{2} = \dfrac{3}{2}, \\ \dfrac{4x}{3} + y = 5. \end{cases}$
11. $\begin{cases} y - \dfrac{x}{3} = \dfrac{1}{6}, \\ 3y - \dfrac{4x}{3} = \dfrac{1}{3}. \end{cases}$
12. $\begin{cases} 2x + 3y = -\dfrac{14}{3}, \\ \dfrac{x}{2} - y = \dfrac{7}{3}. \end{cases}$
13. $\begin{cases} 3x + 5y = 9, \\ 10y - 7x = -8. \end{cases}$
14. $\begin{cases} 4x - 8y = -3, \\ 11x + 5y = -15. \end{cases}$
15. $\begin{cases} 2x - y = 8, \\ 7x + 4y = 43. \end{cases}$

Proceed with the solution until you recognize that the equations are inconsistent or dependent. Then check by graphing the equations.

16. $\begin{cases} x - 2y + 3 = 0, \\ 2x - 4y = 5. \end{cases}$
17. $\begin{cases} x - 2y + 5 = 0, \\ 2x - 4y = -10. \end{cases}$
18. $\begin{cases} 2x - 3y = 5, \\ 4x - 7 = 6y. \end{cases}$
19. $\begin{cases} 4x + 3y = 8, \\ 8x + 6y = 23. \end{cases}$
20. $\begin{cases} 3x - y = 7, \\ 6x = 2y + 14. \end{cases}$
21. $\begin{cases} 5x - 2y - 3 = 0, \\ 10x = 4y + 6. \end{cases}$

100. Elimination by substitution

EXAMPLE 1. Solve for x and y:
$$\begin{cases} 4x + 5y = 6, & (1) \\ 2x + 3y = 4. & (2) \end{cases}$$

SOLUTION. 1. Solve (2) for x:
$$2x = 4 - 3y; \quad x = \tfrac{1}{2}(4 - 3y). \tag{3}$$

2. Substitute $x = \frac{1}{2}(4 - 3y)$ in (1):
$$4[\tfrac{1}{2}(4 - 3y)] + 5y = 6. \tag{4}$$

In obtaining equation 4, we have eliminated x by substituting for x from one given equation into the other.

3. Solve (4) for y:
$$8 - 6y + 5y = 6; \quad y = 2.$$

4. Substitute $y = 2$ in (3):
$$x = \tfrac{1}{2}(4 - 6) = -1.$$

Hence, the solution of the system is $(x = -1, y = 2)$.

SUMMARY. *To solve a system of two linear equations by elimination by substitution:*

1. *Solve one equation for one unknown in terms of the other and substitute the result in the other equation.*

2. *Solve the equation obtained in Step 1 for the second unknown.*

3. *Substitute the value of the second unknown in any equation involving both unknowns and find the value of the first unknown.*

EXERCISE 50

Solve by elimination by substitution and check.

1. $\begin{cases} x = 3y - 1, \\ 2x - 3y = 4. \end{cases}$ **2.** $\begin{cases} 2x + y = -3, \\ 4x - y = 15. \end{cases}$ **3.** $\begin{cases} u = 2v + 4, \\ 10v - 2u = 1. \end{cases}$

4. $\begin{cases} u = -w + 1, \\ 3u + 2w = 0. \end{cases}$ **5.** $\begin{cases} 2y - x = 0, \\ 5x - 4y = 0. \end{cases}$ **6.** $\begin{cases} 4y + 3x + 4 = 0, \\ x + 2y = 0. \end{cases}$

7. $\begin{cases} 2x + 3y = 0, \\ 3x + 5y = 0. \end{cases}$ **8.** $\begin{cases} 2x + 3y = -3, \\ 5x - 2y = 21. \end{cases}$ **9.** $\begin{cases} 3x + 4y = 1, \\ 4x - y = 14. \end{cases}$

10–15. Solve Problems 1–6 of Exercise 49 by substitution.

Clear of fractions if necessary and solve by any method. Do not restrict your choice to just one of the two available methods.

16. $\begin{cases} 3x - 5y = 3, \\ 2x - 7y = 3. \end{cases}$ **17.** $\begin{cases} 6x - 5y = 3, \\ 4y - 9x = 5. \end{cases}$ **18.** $\begin{cases} 9r + 14s = -\frac{11}{2}, \\ 6r + 21s = -7. \end{cases}$

19. $\begin{cases} 3x - 4y = .5, \\ x + 2y = .8. \end{cases}$ **20.** $\begin{cases} 4x + 3y = 6.4, \\ 3x - .5y = 1.5. \end{cases}$ **21.** $\begin{cases} 5y - .6x = 3.45, \\ 4y + 5x = -.67. \end{cases}$

22. $\begin{cases} \frac{1}{2}x = 2 - \frac{5}{4}y, \\ \frac{1}{6}x = \frac{3}{2} + \frac{5}{3}y. \end{cases}$ **23.** $\begin{cases} \frac{3}{2}x - \frac{4}{3}y = -1, \\ \frac{2}{3}x - \frac{1}{4}y = \frac{7}{12}. \end{cases}$ **24.** $\begin{cases} \frac{2}{5}x - \frac{5}{6}y = -\frac{1}{2}, \\ \frac{1}{6}x + \frac{5}{9}y = \frac{5}{2}. \end{cases}$

25. $\begin{cases} \frac{2}{5}x + \frac{5}{6}y + \frac{1}{2} = 0, \\ \frac{1}{6}x - \frac{5}{9}y - \frac{5}{2} = 0. \end{cases}$ **26.** $\begin{cases} \frac{3}{2}x + \frac{4}{3}y + 1 = 0, \\ \frac{2}{3}x + \frac{1}{4}y = \frac{7}{12}. \end{cases}$

27. $\begin{cases} \dfrac{3x - 2y + 3}{2x + 5y} - \dfrac{1}{2} = 0, \\[2ex] \dfrac{x - 3y + 6}{2x + y} - \dfrac{3}{8} = 0. \end{cases}$ **28.** $\begin{cases} \dfrac{2x + 5y + 17}{3x + 2y} + \dfrac{1}{2} = 0, \\[2ex] \dfrac{3x + y + 4}{x + 3y} + \dfrac{3}{5} = 0. \end{cases}$

29. $\begin{cases} \dfrac{x - 2}{x + 1} = \dfrac{2 + y}{1 + y}, \\[2ex] \dfrac{5 + y}{4 + y} - \dfrac{x - 3}{x - 4} = 0. \end{cases}$ **30.** $\begin{cases} \dfrac{6 - y - 6x}{7y + 9x - 8} - 1 = 0, \\[2ex] \dfrac{3x + y - 2}{2y + 3x - 3} - 1 = 0. \end{cases}$

101. Systems involving literal coefficients

If a system involves other letters than the unknowns, it is usually best to solve by finding each unknown in turn by elimination through addition or subtraction.

EXAMPLE 1. Solve for x and y:
$$\begin{cases} ax + by = e, & (1)\\ cx + dy = f. & (2)\end{cases}$$

SOLUTION. 1. Multiply (1) by d: $\quad adx + bdy = de. \qquad (3)$

2. Multiply (2) by b: $\quad bcx + bdy = bf. \qquad (4)$

3. Subtract, (3) − (4): $\quad x(ad - bc) = de - bf. \qquad (5)$

4. Suppose that $ad - bc \neq 0$ and divide by $ad - bc$ in (5).
$$x = \frac{de - bf}{ad - bc}. \qquad (6)$$

5. By similar steps [multiplying (1) by c and (2) by a] we find y.
$$y = \frac{af - ce}{ad - bc}. \qquad (7)$$

102. Systems linear in the reciprocals of the unknowns

EXAMPLE 1. Solve:
$$\begin{cases} \dfrac{3}{x} + \dfrac{5}{y} = 17, & (1)\\[2ex] \dfrac{2}{x} + \dfrac{1}{y} = 2. & (2)\end{cases}$$

SOLUTION. 1. Multiply (2) by 5: $\quad \dfrac{10}{x} + \dfrac{5}{y} = 10. \qquad (3)$

2. Subtract, (3) − (1): $\quad \dfrac{7}{x} = -7; \quad 7 = -7x; \quad x = -1.$

3. Substitute $x = -1$ in (1): $\quad -3 + \dfrac{5}{y} = 17; \quad y = \dfrac{1}{4}.$

The solution of the system is $(x = -1, y = \frac{1}{4})$.

Comment. Equation 1 is said to be linear in $1/x$ and $1/y$ because, if we let $u = 1/x$ and $v = 1/y$, equation 1 becomes $3u + 5v = 17$. Similarly, equation 2 becomes $2u + v = 2$. In place of the preceding solution, we could first solve for u and v; then their reciprocals would give x and y.

EXERCISE 51

Solve for the literal numbers without first clearing of fractions.

1. $\begin{cases} \dfrac{3}{x} - \dfrac{5}{y} = 17, \\[2ex] \dfrac{2}{x} - \dfrac{1}{y} = 2. \end{cases}$

2. $\begin{cases} \dfrac{5}{x} + \dfrac{9}{y} = 7, \\[2ex] \dfrac{4}{x} + \dfrac{15}{y} = 3. \end{cases}$

3. $\begin{cases} \dfrac{2}{5u} + \dfrac{4}{v} = 1, \\[2ex] \dfrac{3}{u} + \dfrac{5}{2v} = 2. \end{cases}$

4. $\begin{cases} \dfrac{2}{x} + \dfrac{5}{y} = 5, \\ \dfrac{8}{x} - \dfrac{15}{y} = 6. \end{cases}$ 5. $\begin{cases} \dfrac{9}{x} + \dfrac{6}{y} = 14, \\ \dfrac{5}{x} + \dfrac{9}{y} = 4. \end{cases}$ 6. $\begin{cases} \dfrac{10}{u} + \dfrac{9}{v} = 14, \\ \dfrac{4}{u} + \dfrac{15}{2v} = 3. \end{cases}$

Solve for x and y, or for w and z.

7. $\begin{cases} ax - 2y = 2 + b, \\ ax + 4y = 2 - 2b. \end{cases}$ 8. $\begin{cases} 2cx - dy = c^2 + d^2, \\ 2x + y = 2c. \end{cases}$

9. $\begin{cases} 2bx + ay = a + b, \\ 2abx - aby = a^2 - b^2. \end{cases}$ 10. $\begin{cases} bw - az - b^2 = 0, \\ aw + bz - bw = a^2. \end{cases}$

11. $\begin{cases} 3hx + y = h, \\ 2kx - 3y = k. \end{cases}$ 12. $\begin{cases} ax + by = 3, \\ bx + ay = 3. \end{cases}$ 13. $\begin{cases} 2aw + bz = ab, \\ w - bz = 3ab + 2b. \end{cases}$

14. $\begin{cases} 2aw + 2bz = 4a^2 + b^2, \\ w - 2z = 2a - b. \end{cases}$ 15. $\begin{cases} aw + bz = a^2 + b^2, \\ bw - az = a^2 + b^2. \end{cases}$

16. $\begin{cases} \dfrac{3x}{2b} - \dfrac{3y}{a} = \dfrac{2a}{b} - \dfrac{b}{a}, \\ \dfrac{x}{2a} + \dfrac{y}{b} = 1. \end{cases}$ 17. $\begin{cases} \dfrac{2m}{2n + 3y} - \dfrac{n}{m - x} = 0, \\ \dfrac{m}{n + x} - \dfrac{2n}{2m - 3y} = 0. \end{cases}$

103. Solution of a system of three linear equations

A system of three linear equations in three unknowns usually has one and only one solution. In special cases, however, such a system may have no solution, in which case the equations are called *inconsistent*, or infinitely many solutions, in which case the equations are called *dependent*. Such cases will not be considered in this book.*

EXAMPLE 1. Solve for x, y, and z:

$$\begin{cases} 3x + y - z = 11, & (1) \\ x + 3y - z = 13, & (2) \\ x + y - 3z = 11. & (3) \end{cases}$$

SOLUTION. 1. Subtract, (1) − (2): $2x - 2y = -2.$ (4)
Multiply (2) by 3: $3x + 9y - 3z = 39.$ (5)
Subtract, (5) − (3): $2x + 8y = 28.$ (6)

2. Solve (4) and (6) for x and y:
Subtract, (6) − (4): $10y = 30; \quad y = 3.$
Substitute $y = 3$ in (4): $2x - 6 = -2; \quad x = 2.$

3. Substitute ($x = 2$, $y = 3$) in (1): $6 + 3 - z = 11; \quad z = -2.$
The solution of the given system is ($x = 2$, $y = 3$, $z = -2$).

* For a more complete treatment, see *College Algebra, Third Edition,* by WILLIAM L. HART; D. C. HEATH AND COMPANY.

SUMMARY. *To solve a system of three linear equations in three unknowns:*

1. *From one pair of the equations, eliminate one of the unknowns; eliminate this unknown from another pair of the original equations.*

2. *Solve the resulting equations for the two unknowns in them.*

3. *Substitute the values of the unknowns found in Step 2 in the simplest of the given equations and solve for the third unknown.*

★*Note 1.* To solve a system of four linear equations in four unknowns, first we would obtain three equations in three of the unknowns by eliminating the other unknown from three different pairs of the original equations. Then, we would solve the new system of three equations and, later, obtain the value of the fourth unknown. A similar but more complicated method would apply to systems in five or more unknowns. A more elegant method is presented in advanced college algebra.

EXERCISE 52

Solve. Do not commence by clearing of fractions.

1. $\begin{cases} 3y - 5x = 1, \\ 3x + z = 1, \\ z + 2y = 2. \end{cases}$

2. $\begin{cases} 2x + y = 2, \\ 2y - 5z = 7, \\ 6x + 2z = 1. \end{cases}$

3. $\begin{cases} 9x + 7 = 5z, \\ 9x - 5y = 3, \\ 3y + 3z = 2. \end{cases}$

4. $\begin{cases} 12x - 4y + z = 3, \\ x - y - 2z = -1, \\ 5x - 2y = 0. \end{cases}$

5. $\begin{cases} 4a + 5b + 6c = 14, \\ 4c - b = 2 + 3a, \\ 14c - 10a - 9b = 10. \end{cases}$

6. $\begin{cases} x - y + 6z = 7, \\ 2x + 3y + 6z = 0, \\ x + 2y + 9z = 3. \end{cases}$

7. $\begin{cases} 2A - 3B - C = 0, \\ 2A + 3B - 2C + 1 = 0, \\ A + 3B + 2C = 1. \end{cases}$

8. $\begin{cases} 2x - y + 2z = 2, \\ 12x + y - 3z = 3, \\ 6x - y + 6z = 12. \end{cases}$

9. $\begin{cases} 9x + 4y + 3z = 3, \\ y - 6x - 6z + 3 = 0, \\ z - y - x = 2. \end{cases}$

10. $\begin{cases} \frac{2}{x} + \frac{1}{y} + \frac{3}{z} + 10 = 0, \\ \frac{2}{x} - \frac{2}{y} + \frac{1}{z} = 2, \\ \frac{6}{x} + \frac{2}{y} - \frac{2}{z} = 5. \end{cases}$

11. $\begin{cases} \frac{2}{x} - \frac{1}{y} - \frac{3}{z} = 7, \\ \frac{1}{x} + \frac{2}{y} - \frac{1}{z} = 10, \\ \frac{3}{x} - \frac{3}{y} + \frac{2}{z} = -7. \end{cases}$

HINT for Problem 10. Let $\frac{1}{x} = u$, $\frac{1}{y} = v$, and $\frac{1}{z} = w$ in all equations.

★12. $\begin{cases} 9u + 5x - y = 6, \\ 4x + 4y + z = -10, \\ 5y + 5z - 3u = -3. \\ 4x - y = -2. \end{cases}$ ★13. $\begin{cases} x + 2y + u = 4, \\ x + 2z - 3u = 6, \\ 3y - x - 6z = -2, \\ 3u + y - z = -2. \end{cases}$

104. Applications of systems of linear equations

EXAMPLE 1. The sum of the digits of a certain two-digit number is 9. If the digits are reversed, the new number is 9 less than 3 times the original number. Find the given number.

SOLUTION. 1. Let t be the tens' digit and u be the units' digit of the number. Then, the number is $10t + u$.

2. If the digits are reversed, u becomes the tens' digit and t the units' digit. The new number is $10u + t$.

3. From the problem, $\begin{cases} u + t = 9, \\ 10u + t = 3(10t + u) - 9. \end{cases}$

We obtain $t = 2$ and $u = 7$. The original number is 27.

EXAMPLE 2. Workmen A and B complete a job if A works 2 days and B works 3 days, or if both work $2\frac{2}{5}$ days. How long would it take each to do the job alone?

SOLUTION. 1. Let x be the number of days it takes A to do the job alone, and y the number of days for B alone.

2. The fractional part of the work which A does in one day is $1/x$ and, for B, is $1/y$. Since they do the whole job, under each set of given data,

$$\frac{2}{x} + \frac{3}{y} = 1, \quad \text{and} \quad \frac{12}{5} \cdot \frac{1}{x} + \frac{12}{5} \cdot \frac{1}{y} = 1.$$

On solving the system consisting of the preceding equations by the method of Section 102, we find $x = 4$ and $y = 6$.

Any line on an (x, y) coordinate plane which is *not parallel to the y-axis* has an equation of the form $y = mx + b$, where m and b are constants. We can use this fact to obtain the equation of a line through two given points.

ILLUSTRATION 1. To find the equation of the line through (4, 3) and (6, 9), we substitute each pair of coordinates for (x, y) in $y = mx + b$:

(when $x = 4$ and $y = 3$) $\quad 3 = 4m + b, \quad$ (1)
(when $x = 6$ and $y = 9$) $\quad 9 = 6m + b. \quad$ (2)

The solution of [(1), (2)] is $(m = 3, b = -9)$. Hence, the equation of the desired line is $y = 3x - 9$. It can be verified by graphing or by substitution that the given points lie on the graph of this equation.

EXERCISE 53

Solve by introducing two or more unknowns.

1. One angle of a triangle is 30° and a second angle is four times the third angle. Find the unknown angles of the triangle.

2. The width of a rectangle exceeds its length by 5 feet and the perimeter of the rectangle is 25 feet. Find the dimensions.

3. Two angles are complementary and one exceeds the other by 7°. Find the angles.

4. A contractor has a daily payroll of $73 when he employs some men at $6 per day and the rest of his workers at $5 per day. If he should double the number receiving $5 and halve the number receiving $6 per day, his daily payroll would be $74. How many employees does he have?

5. How much of a 20% solution of alcohol and how much of a 50% solution should be mixed to give 8 gallons of a 30% solution?

6. How much milk containing 2% butterfat and how much containing 6% butterfat should be mixed to form 100 gallons of milk containing 3% butterfat?

7. If each dimension of a rectangle were increased by 5 feet, the area would be increased by 95 square feet and one dimension would become twice the other. Find the original dimensions.

8. If a two-digit number is divided by its units' digit, the result is 16. If the digits of the given number are reversed, the new number is 18 less than the original one. Find this number.

9. A weight of 5 pounds is 6 feet from the fulcrum on the right-hand side of a lever. It is balanced if we place a first weight 4 feet from the fulcrum on the right and a second weight 7 feet from the fulcrum on the left, or if we place the first weight 8 feet to the right and the second weight 9 feet to the left of the fulcrum. Find the unknown weights.

10. If we seat a boy at 5 feet and a girl at 8 feet from the fulcrum on one side of a teeterboard, they balance a man weighing 160 pounds who is seated 6 feet from the fulcrum on the other side. Balance is maintained if the boy moves to $8\frac{1}{2}$ feet and the girl to 4 feet from the fulcrum on their side. Find their weights.

11. When we divide a certain two-digit number by its tens' digit, the result is 13. If we reverse digits in the number and then divide by the original number, the result is 31/13. Find the original number.

12. The sum of the reciprocals of two numbers is 1/2 and the difference of the reciprocals is 1/6. Find the numbers.

13. How much silver and lead should be added to 100 pounds of a mixture containing 15% silver and 30% lead to obtain an alloy containing 25% silver and 50% lead?

14. How much chromium and nickel should be added to 100 pounds of an alloy containing 5% chromium and 40% nickel to give an alloy containing 15% chromium and 50% nickel?

15. A man divides $10,000 among three investments, at 3%, 4%, and 6% per annum, respectively. His annual income from the first two investments is $80 less than his income from the third investment and his total income is $460 per year. Find the amount invested at each rate.

16. Workmen A and B complete a certain job if they work together for 6 days or if A alone works for 3 days and B alone works for 10 days. How long does it take each man to complete the job alone?

17. In a three-digit number which is 31 times the sum of the digits, the units' digit is one half the sum of the other digits. If the digits are reversed, the new number obtained is 99 greater than the original number. Find its digits.

Find the equation of the line through the given points on an (x, y) coordinate system by solving a pair of equations for two unknowns, or by inspection.

18. $(2, -3)$; $(4, 3)$.

19. $(-3, 1)$; $(-2, -3)$.

20. $(3, -2)$; $(-3, -12)$.

21. $(-4, 5)$; $(8, -4)$.

22. $(-3, 5)$; $(-3, -2)$.

23. $(4, -2)$; $(9, -2)$.

24. An airplane, flying with the wind, took 2.5 hours for a 625-mile run and took 4 hours and 10 minutes to return against the same wind. Find the velocity of the wind and the speed of the airplane in calm air.

25. An army messenger will travel at a speed of 60 miles per hour on land and in a motorboat whose speed is 20 miles per hour in still water. In delivering a message he will go by land to a dock on a river and then on the river against a current of 4 miles per hour. If he reaches his destination in $4\frac{1}{2}$ hours and then returns to his starting point in $3\frac{1}{2}$ hours, how far did he travel by land and how far by water?

CHAPTER 9

EXPONENTS AND RADICALS

105. Proofs of the index laws

We have already employed the following results, called *index laws*, which govern the use of positive integral exponents.

I. *Law of exponents for multiplication:* $a^m a^n = a^{m+n}$.

Proof. 1. By definition, $a^m = a \cdot a \cdots a;$ (m factors)

$a^n = a \cdot a \cdot a \cdots a.$ (n factors)

2. Hence, $a^m a^n = (a \cdot a \cdots a)(a \cdot a \cdot a \cdots a) = a^{m+n}.$
[($m + n$) factors a]

II. *Law for finding a power of a power:* $(a^m)^n = a^{mn}$.

Proof. 1. $(a^m)^n = a^m \cdot a^m \cdots a^m;$ (n factors a^m)

(By Law I) $= a^{m+m+\cdots+m}.$ (n terms m)

2. Since $(m + m + \cdots + m)$ to n terms equals mn, $(a^m)^n = a^{mn}$.

III. *Laws of exponents for division:*

$$\frac{a^m}{a^n} = a^{m-n} \text{ (if } m > n\text{)}; \qquad \frac{a^m}{a^n} = \frac{1}{a^{n-m}} \text{ (if } n > m\text{)}.$$

ILLUSTRATION 1. $\dfrac{a^8}{a^3} = a^5.$ $\dfrac{a^7}{a^{10}} = \dfrac{1}{a^3}.$

Proof, for the case $n > m$. By the definition of a^m and a^n,

$$\frac{a^m}{a^n} = \frac{\cancel{a} \cdot \cancel{a} \cdots \cancel{a}}{a \cdot a \cdots a \cdot \cancel{a} \cdot \cancel{a} \cdots \cancel{a}}; \qquad \begin{matrix}(m \text{ factors})\\(n \text{ factors})\end{matrix}$$

[($n - m$) factors a] $$= \frac{1}{a \cdot a \cdots a} = \frac{1}{a^{n-m}}.$$

IV. *Law for finding a power of a product:* $(ab)^n = a^nb^n$.

Proof. $(ab)^n = ab \cdot ab \cdots ab;$ (n factors ab)

(n factors a, and b) $= (a \cdot a \cdots a)(b \cdot b \cdots b) = a^nb^n.$

Law IV extends to products of any number of factors. Thus,

$$(abc)^n = a^nb^nc^n.$$

ILLUSTRATION 2. $(4a^2b^4)^3 = 4^3(a^2)^3(b^4)^3 = 64a^6b^{12}.$

V. *Law for finding a power of a quotient:* $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}.$

ILLUSTRATION 3. $\left(\dfrac{a}{b}\right)^4 = \dfrac{a^4}{b^4}. \qquad \left(\dfrac{a^2}{3b^3}\right)^2 = \dfrac{(a^2)^2}{(3b^3)^2} = \dfrac{a^4}{9b^6}.$

Proof. $\left(\dfrac{a}{b}\right)^n = \dfrac{a}{b} \cdot \dfrac{a}{b} \cdots \dfrac{a}{b};$ $\left(n \text{ factors } \dfrac{a}{b}\right)$

(n factors a)
(n factors b) $= \dfrac{a \cdot a \cdots a}{b \cdot b \cdots b} = \dfrac{a^n}{b^n}.$

Note 1. The determination of powers of numbers is called **involution.**

EXERCISE 54

Find each power by use of the definition of an exponent.

1. 2^5. **2.** $(-5)^4$. **3.** $(-3)^5$. **4.** $(.1)^4$. **5.** $(\frac{3}{4})^2$. **6.** $(-\frac{2}{3})^3$.

7. What is the sign of an odd power of a negative number?

Perform the operations by use of the index laws.

8. a^5a^8. **9.** x^4x^c. **10.** 2^32^m. **11.** $(x^3)^5$.

12. $(3x)^4$. **13.** $(2a^2)^5$. **14.** $(4b^3)^3$. **15.** $(5x^2y)^4$.

16. dd^4d^6. **17.** $(-2x^2)^3$. **18.** $(-by^4)^3$. **19.** $(-2a^3)^4$.

20. $(b^3)^n$. **21.** $(a^k)^2$. **22.** $(c^h)^{3n}$. **23.** $(d^2)^{hk}$.

24. $(a^mb^2)^n$. **25.** $(c^nd^3)^k$. **26.** $(-.2a^2b)^3$. **27.** $(-.3cd^3)^2$.

28. $\dfrac{x^{12}}{x^4}$. **29.** $\dfrac{z^4}{z^9}$. **30.** $\left(\dfrac{x}{y}\right)^5$. **31.** $\left(\dfrac{c}{d}\right)^3$. **32.** $\left(\dfrac{3}{2x}\right)^4$.

33. $\left(\dfrac{3y}{2}\right)^2$. **34.** $\dfrac{x^2y^8}{x^4y^4}$. **35.** $\dfrac{a^5b^7}{a^3b^9}$. **36.** $\dfrac{a^{16}y^5}{a^3y^3}$. **37.** $\dfrac{ab^7}{a^4b^3}$.

38. $\left(\dfrac{2a}{5b^3}\right)^2$. **39.** $\left(\dfrac{4a^2}{3x}\right)^3$. **40.** $\left(\dfrac{a^2}{b}\right)^h$. **41.** $\left(\dfrac{c^3}{d^2}\right)^k$. **42.** $\left(\dfrac{3a}{b^2}\right)^n$.

43. $\left(\dfrac{z^n}{a^h}\right)^3$. **44.** $\left(\dfrac{-5}{x^2}\right)^4$. **45.** $\left(\dfrac{ab^2}{-3}\right)^3$. **46.** $\left(\dfrac{a^xb}{y^n}\right)^h$. **47.** $\left(\dfrac{c^xd^y}{a^2}\right)^n$.

48. $\left(\dfrac{c}{2d}\right)^2\left(\dfrac{4d}{3b}\right)^3$. **49.** $\left(\dfrac{2x}{3y}\right)^2\left(\dfrac{3yz}{4x}\right)^4$. **50.** $\left(\dfrac{-2x}{3ab}\right)^3\left(\dfrac{4ax}{b}\right)^2$.

51. $\left(\dfrac{5zw}{2xy}\right)^2\left(-\dfrac{2}{w^2z}\right)^3$. **52.** $\left(\dfrac{-2x}{-yz}\right)^5\left(\dfrac{3z}{4x}\right)^2$. **53.** $\left(\dfrac{2y}{5z}\right)^2\left(\dfrac{-zw}{-2x}\right)^5$.

54. Prove that part of Law III which applies if $m > n$.

55. (*a*) Compute $(-2)^4$ and -2^4. (*b*) Under what condition on the positive integer n will $(-3)^n = -3^n$?

106. Imaginary numbers

We have called R a *square root* of A if $R^2 = A$. If A is positive, it has exactly *two square roots,* one positive and one negative, denoted by $\pm\sqrt{A}$.

ILLUSTRATION 1. The square roots of 4 are $\pm\sqrt{4}$ or ± 2.

If a negative number $-P$ has R as a square root, then $R^2 = -P$. But, if R is either positive or negative then R^2 is positive and thus cannot equal $-P$. Hence, $-P$ has *no positive or negative square root.* Therefore, in order that $-P$ may have square roots, we *define* the symbol $\sqrt{-P}$ as a new variety of number, called an **imaginary number,** with the property that

$$(\sqrt{-P})^2 = -P \qquad \text{and} \qquad (-\sqrt{-P})^2 = -P.$$

Thus, $-P$ has the two imaginary numbers $\pm\sqrt{-P}$ as square roots. As an immediate extension of the preceding terminology, we agree that, if M is a real number, each of the expressions $(M + \sqrt{-P})$ and $(M - \sqrt{-P})$ will be called an imaginary number.

ILLUSTRATION 2. The square roots of the negative number -5 are the imaginary numbers $\pm\sqrt{-5}$. $(7 + \sqrt{-18})$ is an imaginary number.

Unless otherwise stated, any literal number in this book will represent a *real* number. Imaginary numbers will not enter actively into our discussions until we meet them in the solution of equations in a later chapter.

Note 1. The somewhat unfortunate name *imaginary number* is inherited from a time when mathematicians actually considered such a number to be imaginary in the colloquial sense. In a similar fashion, our common negative numbers, at their first introduction into mathematics, were also called illusory or fictitious. The student will soon appreciate that imaginary numbers deserve consideration on an equal footing with real numbers. Imaginary numbers are indispensable not only in pure mathematics but also in important fields where mathematics is applied. Imaginary numbers will be studied in more detail in a later chapter.

107. Roots

We call R a *square root* of A if $R^2 = A$ and a *cube root* of A if $R^3 = A$. If n is any positive integer we say that

$$\textbf{R is an nth root of A if } R^n = A. \qquad (1)$$

ILLUSTRATION 1. The only nth root of 0 is 0. 2 is a 5th root of 32 because $2^5 = 32$. -3 is a cube root of -27.

The following facts are proved in college algebra when imaginary numbers are treated in a complete fashion.

1. *Every number A, not zero, has just n distinct* nth *roots, some or all of which may be imaginary numbers.*

2. *If n is* **even,** *every positive number A has just* **two** *real* nth *roots, one positive and one negative, with equal absolute values.*

3. *If n is* **odd,** *every real number A has just* **one** *real* nth *root, which is positive when A is positive and negative when A is negative.*

4. *If n is* **even** *and A is* **negative,** *all* nth *roots of A are imaginary numbers.*

If A is *positive,* its *positive* nth root is called the **principal *n*th root** of A. If A is negative and n is odd, the *negative* nth root of A is called its *principal* nth *root.*

ILLUSTRATION 2. The real 4th roots of 81 are ± 3 and $+3$ is the principal 4th root. The principal cube root of $+125$ is $+5$ and of -125 is -5. All 4th roots of -16 are imaginary numbers.

ILLUSTRATION 3. The real cube root of 8 is 2. Also, by advanced methods it can be shown that 8 has the two imaginary cube roots $(-1 + \sqrt{-3})$ and $(-1 - \sqrt{-3})$.

108. Radicals

The *radical* $\sqrt[n]{A}$, which we read *the* nth root of A, is used to denote the *principal* nth root of A when it has a real nth root, and to denote any convenient * nth root of A if all nth roots are imaginary. In $\sqrt[n]{A}$, the positive integer n is called the *index* or *order* of the radical, and A is called its **radicand.** When $n = 2$, we omit writing the index and use $\sqrt{A}$ instead of $\sqrt[2]{A}$ for the square root of A.

I. $\sqrt[n]{A}$ *is positive if* A *is positive.*

II. $\sqrt[n]{A}$ *is negative if* A *is negative and* n *is odd.*

III. $\sqrt[n]{A}$ *is imaginary if* A *is negative and* n *is even.*

ILLUSTRATION 1. $\sqrt[4]{81} = 3$. $\sqrt[5]{-32} = -2$. $\sqrt[4]{-8}$ is imaginary.

ILLUSTRATION 2. $\sqrt[4]{\frac{1}{81}} = \frac{1}{3}$ because $\left(\frac{1}{3}\right)^4 = \frac{1}{3^4} = \frac{1}{81}$.

By the definition of an nth root,

$$(\sqrt[n]{A})^n = A. \tag{1}$$

ILLUSTRATION 3. $(\sqrt{3})^2 = 3$. $\sqrt[7]{(169)^7} = 169$. $(\sqrt[5]{2cd^2})^5 = 2cd^2$.

$$(\sqrt[4]{c^2 + cd + d^2})^4 = c^2 + cd + d^2.$$

Note 1. To avoid ambiguous signs and imaginary numbers in elementary problems, the following agreement will hold in this book unless otherwise specified. If the index of a radical is an *even* integer, all literal numbers in the radicand not used as exponents represent *positive numbers*, and are such that the radicand is *positive*.

By the definition of $\sqrt[n]{A}$ as a principal nth root, it follows that, under the agreement † of Note 1,

$$\sqrt[n]{a^n} = a. \tag{2}$$

ILLUSTRATION 4. $\sqrt[3]{x^3} = x$. $\sqrt[4]{5^4} = 5$.

* This matter of convenience is discussed in more advanced treatments of imaginary numbers. If all nth roots of A are imaginary, it is not usual to call any particular one of them the *principal* nth root.

† If a is negative and n is even, then a^n is *positive* and the *positive* nth root of a^n is $-a$, or $\sqrt[n]{a^n} = -a$. This case is ruled out by Note 1 and does not come under formula 2. For instance, if a is negative, $\sqrt{a^2} = -a$.

EXERCISE 55

State the two square roots of each number.

1. 64. 2. 49. 3. 81. 4. 121. 5. $\frac{1}{9}$. 6. $\frac{4}{25}$. 7. .01.

State the principal square root of each number.

8. 16. 9. 144. 10. 100. 11. $\frac{1}{36}$. 12. $\frac{1}{64}$. 13. $\frac{1}{49}$.

State the principal cube root of each number.

14. 8. 15. -27. 16. 27. 17. 125. 18. -1. 19. -216. 20. $-\frac{1}{27}$.

State the two real fourth roots of each number.

21. 81. 22. 16. 23. 625. 24. 10,000. 25. $\frac{1}{16}$. 26. .0001.

Find the specified power of the radical, or the indicated root.

27. $\sqrt{d^2}$. 28. $\sqrt[3]{y^3}$. 29. $\sqrt[4]{3^4}$. 30. $(\frac{1}{2})^5$.

31. $\sqrt[6]{3^6}$. 32. $(\sqrt{17})^2$. 33. $(\sqrt[4]{57})^4$. 34. $(\sqrt[3]{-19})^3$.

35. $(\sqrt[5]{4xy^3})^5$. 36. $\sqrt{25}$. 37. $\sqrt{36}$. 38. $\sqrt[3]{8}$.

39. $\sqrt[3]{-8}$. 40. $\sqrt[3]{-125}$. 41. $\sqrt[5]{32}$. 42. $\sqrt[4]{81}$.

43. $\sqrt[3]{64}$. 44. $\sqrt[5]{243}$. 45. $\sqrt[4]{16}$. 46. $\sqrt[3]{-1}$.

47. $\sqrt[5]{-1}$. 48. $\sqrt[7]{1}$. 49. $\sqrt[3]{216}$. 50. $\sqrt[4]{625}$.

51. $\sqrt{400}$. 52. $\sqrt{900}$. 53. $\sqrt[3]{8000}$. 54. $\sqrt[3]{-64}$.

55. $\sqrt[3]{\frac{1}{8}}$. 56. $\sqrt[4]{\frac{1}{81}}$. 57. $\sqrt[3]{-\frac{1}{27}}$. 58. $\sqrt[3]{-\frac{1}{64}}$.

59. $\sqrt{.01}$. 60. $\sqrt[3]{.008}$. 61. $\sqrt[3]{.001}$. 62. $\sqrt[3]{.027}$. 63. $\sqrt[4]{.0016}$.

109. Rational and irrational numbers

A real number which can be expressed as a fraction m/n, where the numerator and denominator are integers, is called a *rational number.* All integers are included among the rational numbers because, if m is any integer, then m can be expressed as the fraction $m/1$. A real number which is *not* a rational number is called an *irrational number.*

ILLUSTRATION 1. 7, 0, and $-\frac{5}{8}$ are rational numbers. Any terminating decimal fraction is a rational number. Thus, 3.017, or 3017/1000, is a rational number. π and $\sqrt{2}$ are irrational numbers. A proof of the irrationality of $\sqrt{2}$ is given in the Appendix, Note 1. Any irrational number can be expressed as an endless but not a repeating decimal. Thus, $\pi = 3.14159 \cdots$ and $\sqrt{2} = 1.414 \cdots$ are endless but not repeating decimals.

If A is not the nth power of a rational number, and $\sqrt[n]{A}$ is real, then $\sqrt[n]{A}$ is irrational and is called a **surd** of the nth order.

ILLUSTRATION 2. $\sqrt{3}$ is a surd. $\sqrt[6]{64}$ is not a surd because $\sqrt[6]{64} = 2$. A surd of the second order is sometimes called a *quadratic surd* and one of the third order a *cubic surd*. The values of various quadratic and cubic surds are given to a limited number of decimal places in Table I.

110. Rational and irrational expressions

An algebraic expression is said to be *rational* in certain letters if it can be expressed as a fraction whose numerator and denominator are *integral rational polynomials* in the letters. An algebraic expression which is *not* rational in the letters is said to be *irrational* in them.

ILLUSTRATION 1. $\dfrac{x^3 - 3a^2 + 1}{2x - 5a}$ is rational in a and x.

ILLUSTRATION 2. $\sqrt{3x + y}$ is not rational in x and y.

ILLUSTRATION 3. The expression $x\sqrt{2} - 3x^2 - \sqrt{5}$ is an integral rational polynomial in x. The presence of irrational explicit numbers, $\sqrt{2}$ and $\sqrt{5}$, is of no concern.

Hereafter, unless otherwise stated, in any integral rational polynomial we shall assume that the numerical coefficients are *rational* numbers.

111. Perfect powers of rational functions

A rational expression is called a *perfect* nth *power* if it is the nth power of some rational expression. Also, we say that a rational number is a perfect nth power if it is the nth power of some *rational* number.

If an integral rational term is a perfect nth power, the numerical coefficient separately is a perfect nth power. Also, each exponent in the term has n as a factor, because in obtaining the nth power of a term each exponent is multiplied by n. In verifying that a term is a perfect nth power, first factor the coefficient.

ILLUSTRATION 1. $32y^{15}$ is a perfect 5th power: $\quad 32y^{15} = 2^5y^{15} = (2y^3)^5.$

ILLUSTRATION 2. $\dfrac{x^2}{9b^6}$ is a perfect square: $\quad \dfrac{x^2}{9b^6} = \left(\dfrac{x}{3b^3}\right)^2.$

112. Elementary properties of radicals

The following Properties I and II have already been met in Section 108 as direct consequences of the definition of an nth root.

I. $(\sqrt[n]{a})^n = a.$

II. $\sqrt[n]{a^n} = a.$ (*If a is positive when n is even*)

III. $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}.$

ILLUSTRATION 1. $\sqrt{9x^2} = \sqrt{9}\sqrt{x^2} = 3x$, because $(3x)^2 = 9x^2$.

ILLUSTRATION 2. $\sqrt[3]{ab} = \sqrt[3]{a}\sqrt[3]{b}$ because

$$(\sqrt[3]{a}\sqrt[3]{b})^3 = (\sqrt[3]{a})^3(\sqrt[3]{b})^3 = ab.$$

IV. $$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}.$$

ILLUSTRATION 3. $\sqrt[4]{\frac{81}{16}} = \frac{\sqrt[4]{81}}{\sqrt[4]{16}} = \frac{3}{2}$, because $\left(\frac{3}{2}\right)^4 = \frac{3^4}{2^4} = \frac{81}{16}.$

ILLUSTRATION 4. $\sqrt[4]{\frac{a}{b}} = \frac{\sqrt[4]{a}}{\sqrt[4]{b}}$, because $\left(\frac{\sqrt[4]{a}}{\sqrt[4]{b}}\right)^4 = \frac{(\sqrt[4]{a})^4}{(\sqrt[4]{b})^4} = \frac{a}{b}.$

V. *If m/n is an integer,* $\sqrt[n]{a^m} = a^{\frac{m}{n}}.$

ILLUSTRATION 5. $\sqrt[3]{a^{12}} = a^{\frac{12}{3}} = a^4$, because $(a^4)^3 = a^{12}$.

★*Note 1.* The following proofs are complete if a and b are *positive*, because then all principal roots involved are positive. The interested student may consider the possibility of negative values for a and b. To complete the proofs, it should be demonstrated that in all cases the two sides of each formula in (III), (IV), and (V) are either both positive or both negative and hence are equal.

Proof of (III). Raise $(\sqrt[n]{a}\sqrt[n]{b})$ to the nth power:

$$(\sqrt[n]{a}\sqrt[n]{b})^n = (\sqrt[n]{a})^n(\sqrt[n]{b})^n = ab.$$

Hence, by the definition of an nth root, $\sqrt[n]{a}\sqrt[n]{b}$ is an nth root of ab, or

$$\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}.$$

Proof of (V). By Law II, page 142, $(a^{\frac{m}{n}})^n = a^{\frac{m}{n}\cdot n} = a^m$. Hence, $a^{\frac{m}{n}}$ is an nth root of a^m, or $a^{\frac{m}{n}} = \sqrt[n]{a^m}$.

We observe that Property II is a special case of Property V, with $m = n$. However, we dignify Property II by special attention because, if we are able to express A as a perfect nth power, Property II gives $\sqrt[n]{A}$ by mere inspection.

ILLUSTRATION 6. $\sqrt[3]{8x^6y^9} = \sqrt[3]{(2x^2y^3)^3} = 2x^2y^3.$ (Property II)

Or, by Properties III and V, $\sqrt[3]{8x^6y^9} = \sqrt[3]{8}\sqrt[3]{x^6}\sqrt[3]{y^9} = 2x^2y^3.$

EXERCISE 56

Each radicand is a perfect power. Find the specified root.

1. $\sqrt{b^2}$. 2. $\sqrt[3]{d^3}$. 3. $\sqrt[4]{a^4}$. 4. $\sqrt[5]{a^5}$. 5. $\sqrt{x^4}$.

6. $\sqrt{y^6}$. 7. $\sqrt{z^8}$. 8. $\sqrt[3]{x^6}$. 9. $\sqrt[3]{y^9}$. 10. $\sqrt[3]{z^{12}}$.

11. $\sqrt[4]{x^8}$. 12. $\sqrt[5]{y^{10}}$. 13. $\sqrt[4]{x^{12}}$. 14. $\sqrt{y^{10}}$. 15. $\sqrt{4y^2}$.

16. $\sqrt{9z^2}$. 17. $\sqrt[3]{8y^3}$. 18. $\sqrt[3]{27x^3}$. 19. $\sqrt{\frac{9}{16}}$. 20. $\sqrt{\frac{25}{49}}$.

21. $\sqrt[3]{\frac{8}{27}}$. 22. $\sqrt[4]{\frac{16}{81}}$. 23. $\sqrt[3]{\frac{27}{1000}}$. 24. $\sqrt[3]{\frac{64}{27}}$. 25. $\sqrt[5]{\frac{243}{32}}$.

26. $\sqrt[4]{16x^4}$. 27. $\sqrt[4]{81x^8}$. 28. $\sqrt[3]{-a^3}$. 29. $\sqrt[3]{-8x^3}$.

30. $\sqrt{y^4w^6}$. 31. $\sqrt[3]{x^6y^9}$. 32. $\sqrt{9x^8}$. 33. $\sqrt[3]{8a^6}$.

34. $\sqrt[3]{\frac{125}{8}}$. 35. $\sqrt[3]{-.001}$. 36. $\sqrt[3]{27y^9}$. 37. $\sqrt[4]{16x^4y^8}$.

38. $\sqrt[4]{81z^{12}}$. 39. $\sqrt[5]{-32z^{10}}$. 40. $\sqrt[4]{.0625}$. 41. $\sqrt[3]{-x^3z^9}$.

42. $\sqrt[3]{216a^6}$. 43. $\sqrt{.04x^{10}}$. 44. $\sqrt[3]{.008y^6}$. 45. $\sqrt[3]{.125x^3}$.

46. $\sqrt[3]{\dfrac{x^3}{8}}$. 47. $\sqrt{\dfrac{4x^2}{9y^4}}$. 48. $\sqrt{\dfrac{25z^6}{4x^4}}$. 49. $\sqrt[3]{\dfrac{27x^6}{8y^9}}$.

113. Fractional powers

We have previously defined a^p only when p is a positive integer. We shall now introduce other types of powers in such a way that all the types, together, will obey laws of the same forms as those for positive integral exponents.

If fractional exponents are to obey the law of exponents for multiplication, then, for example,

$$a^{\frac{1}{2}}a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a, \quad \text{or} \quad (a^{\frac{1}{2}})^2 = a;$$

thus, $a^{\frac{1}{2}}$ must be a square root of a. Similarly, we should have

$$a^{\frac{5}{2}}a^{\frac{5}{2}} = a^{\frac{5}{2}+\frac{5}{2}} = a^5, \quad \text{or} \quad (a^{\frac{5}{2}})^2 = a^5,$$

so that $a^{\frac{5}{2}}$ should be a square root of a^5. Accordingly, if m and n are any positive integers, *we define* $a^{\frac{m}{n}}$ **to be the principal nth root of a^m:**

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}; \tag{1}$$

[when $m = 1$ in (1)] $$a^{\frac{1}{n}} = \sqrt[n]{a}. \tag{2}$$

Thus, we may use *fractional exponents* instead of *radicals* to denote *principal roots.* The defining equation 1 is consistent with Property V of page 149, which was proved for the case where m/n was an integer.

ILLUSTRATION 1. $8^{\frac{1}{3}} = \sqrt[3]{8} = 2.$ $(-8)^{\frac{1}{3}} = \sqrt[3]{-8} = -2.$

ILLUSTRATION 2. $x^{\frac{8}{3}} = \sqrt[3]{x^8}.$ $8^{\frac{2}{3}} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4.$

ILLUSTRATION 3. $(-8)^{\frac{2}{3}} = \sqrt[3]{(-8)^2} = \sqrt[3]{64} = 4.$

In (1), we defined $a^{\frac{m}{n}}$ to be the nth root of a^m. It is very useful to prove the theorem that, also, $a^{\frac{m}{n}}$ is the mth *power of the* nth *root of* a, or

$$a^{\frac{m}{n}} = (\sqrt[n]{a})^m. \tag{3}$$

To prove (3) we must show that the right members of (1) and (3) are identical, or that

$$\sqrt[n]{a^m} = (\sqrt[n]{a})^m. \tag{4}$$

ILLUSTRATION 4. To show that $\sqrt[3]{a^4} = (\sqrt[3]{a})^4$, raise the right member to the 3d power and use the laws for integral exponents:

$$[(\sqrt[3]{a})^4]^3 = (\sqrt[3]{a})^{12} = [(\sqrt[3]{a})^3]^4 = (a)^4 = a^4.$$

Hence, $(\sqrt[3]{a})^4$ is a *cube root* of a^4. If we assume that a is *positive*, then this cube root is *positive* and hence is identical with the *principal* cube root represented by $\sqrt[3]{a^4}$.

In accordance with Note 1 on page 146, we agree *not to deal with the symbol $a^{\frac{m}{n}}$ if n is even and $a < 0$.* With this case eliminated, we prove (4) by raising the right-hand side to the nth power, showing that we obtain a^m, and thus demonstrating that $(\sqrt[n]{a})^m$ is an nth root of a^m:

$$[(\sqrt[n]{a})^m]^n = [(\sqrt[n]{a})^n]^m = [a]^m = a^m.$$

ILLUSTRATION 5. From (3), since $\sqrt[6]{64} = 2$,

$$64^{\frac{5}{6}} = (\sqrt[6]{64})^5 = 2^5 = 32.$$

Notice the relative inconvenience of the following evaluation by use of (1)

$$64^{\frac{5}{6}} = \sqrt[6]{64^5} = \sqrt[6]{(2^6)^5} = \sqrt[6]{2^{30}} = 2^5 = 32.$$

★*Note 1.* The difficulties met if $a^{\frac{m}{n}}$ is used when a is negative and n is eve are illustrated below where a contradiction "$+ 1 = - 1$" results from reck less use of the symbol $(- 1)^{\frac{2}{6}}$.

$$- 1 = \sqrt[3]{- 1} = (- 1)^{\frac{1}{3}} = (- 1)^{\frac{2}{6}} = \sqrt[6]{(- 1)^2} = \sqrt[6]{+ 1} = + 1.$$

114. Zero as an exponent

If operations with a^0 are to obey the law of exponents for multiplica tion, then we should define a^0 so that

$$a^0a^n = a^{0+n} = a^n, \quad \text{or} \quad a^0a^n = a^n, \quad \text{or} \quad a^0 = \frac{a^n}{a^n} = 1.$$

Hence, if $a \neq 0$, *we define* a^0 *by the equation* $\mathbf{a^0 = 1}$.

ILLUSTRATION 1. By definition, $5^0 = 1$. $(17x)^0 = 1$.

115. Negative exponents

If a negative exponent is to obey the law of exponents for mul tiplication, then, for instance, we should have $a^3a^{-3} = a^{3-3} = a^0 = 1$ Hence, if p is any positive exponent of the types previously intro duced, we define a^{-p} by the equation $a^pa^{-p} = 1$, or

$$a^{-p} = \frac{1}{a^p}. \qquad (1$$

ILLUSTRATION 1. $x^{-5} = \dfrac{1}{x^5}$. $5^{-2} = \dfrac{1}{5^2} = \dfrac{1}{25}$. $8^{-\frac{1}{3}} = \dfrac{1}{8^{\frac{1}{3}}} = \dfrac{1}{2}$.

In a fraction, any power which is a factor of one term (numerato or denominator) may be removed if the factor, with *the sign of it exponent changed,* is written as a factor of the other term. That is

$$\frac{a}{bx^n} = \frac{ax^{-n}}{b}. \qquad (2$$

Proof. $\dfrac{a}{bx^n} = \dfrac{a}{b} \cdot \dfrac{1}{x^n} = \dfrac{a}{b} \cdot x^{-n} = \dfrac{ax^{-n}}{b}$.

We may use negative exponents in avoiding fractions.

ILLUSTRATION 2. $$\frac{17a}{b^2} = 17a \cdot \frac{1}{b^2} = 17ab^{-2}.$$

ILLUSTRATION 3. To express the following fraction with positive exponents, we may use (2) mechanically:

$$\frac{3a^{-2}b^3}{c^{-3}a^4} = \frac{3c^3b^3}{a^2a^4} = \frac{3c^3b^3}{a^6}. \tag{3}$$

In more detailed fashion we obtain (3) as follows:

$$\frac{3a^{-2}b^3}{c^{-3}a^4} = \left(3 \cdot \frac{1}{a^2} \cdot b^3\right) \div \left(a^4 \cdot \frac{1}{c^3}\right) = \frac{3b^3}{a^2} \cdot \frac{c^3}{a^4} = \frac{3b^3c^3}{a^6}. \tag{4}$$

The student should act as in (3) but should also appreciate (4).

EXERCISE 57

Find the value of the symbol by changing from a fractional exponent to a radical or from a negative to a positive exponent if necessary.

1. $9^{\frac{1}{2}}$. **2.** $25^{\frac{1}{2}}$. **3.** $8^{\frac{1}{3}}$. **4.** $27^{\frac{1}{3}}$. **5.** 4^{-1}.

6. 2^{-1}. **7.** 35^{-1}. **8.** 16^0. **9.** $4^{\frac{3}{2}}$. **10.** $81^{\frac{1}{4}}$.

11. 3^{-2}. **12.** 5^{-3}. **13.** 3^{-4}. **14.** $16^{\frac{1}{4}}$. **15.** $(\frac{1}{4})^{\frac{1}{2}}$.

16. $(\frac{1}{36})^{\frac{1}{2}}$. **17.** $(\frac{1}{27})^{\frac{1}{3}}$. **18.** $(-8)^{\frac{1}{3}}$. **19.** 7^0. **20.** $16^{-\frac{1}{2}}$.

21. $9^{-\frac{1}{2}}$. **22.** $27^{-\frac{1}{3}}$. **23.** $(\frac{1}{3})^{-1}$. **24.** $(\frac{2}{5})^{-1}$. **25.** $(.36)^{\frac{1}{2}}$.

26. $(.09)^{\frac{1}{2}}$. **27.** $(-3)^{-2}$. **28.** 725^0. **29.** $(-5)^{-2}$. **30.** $(-3)^{-4}$.

31. $(-2)^{-5}$. **32.** $(-5)^{-3}$. **33.** $(-1)^{\frac{1}{3}}$. **34.** $(-8)^{\frac{1}{3}}$. **35.** $(-8)^{-\frac{1}{3}}$.

36. $(-27)^{-\frac{1}{3}}$. **37.** $(-125)^{\frac{1}{3}}$. **38.** $(.0081)^{\frac{1}{4}}$. **39.** $(.0001)^{-\frac{1}{4}}$.

Find the value of the symbol by use of formula 3, *page* 151.

40. $8^{\frac{2}{3}}$. **41.** $25^{\frac{3}{2}}$. **42.** $4^{\frac{5}{2}}$. **43.** $36^{\frac{3}{2}}$. **44.** $81^{\frac{3}{4}}$.

45. $125^{\frac{4}{3}}$. **46.** $(\frac{1}{9})^{\frac{3}{2}}$. **47.** $(\frac{1}{27})^{\frac{2}{3}}$. **48.** $(-27)^{\frac{2}{3}}$. **49.** $(-64)^{\frac{2}{3}}$.

Express with positive exponents.

50. a^{-3}. **51.** b^{-4}. **52.** c^{-6}. **53.** $x^{-2}y$. **54.** $a^{-3}b^2$.

55. c^2d^{-3}. **56.** $6x^{-2}$. **57.** $3h^{-4}$. **58.** $2ab^{-3}$. **59.** $4x^{-3}y$.

60. $x^{-4}y^{-5}$. **61.** $2ay^{-5}$. **62.** $3ax^{-2}y^{-3}$. **63.** $4^{-1}ax^{-3}$. **64.** $5^{-2}cx^{-3}$.

65. $\frac{1}{b^{-4}}$. **66.** $\frac{3}{a^{-5}}$. **67.** $\frac{c}{a^{-3}}$. **68.** $\frac{a^3}{d^{-2}}$. **69.** $\frac{a^{-3}}{5}$.

70. $\dfrac{c^3}{a^{-2}}$. **71.** $\dfrac{a^{-2}b^3}{c^2d^4}$. **72.** $\dfrac{x^{-2}b^3}{5a^{-3}}$. **73.** $\dfrac{cd^{-3}}{6a^{-3}}$. **74.** $\dfrac{3x^{-5}}{2a^{-3}}$.

75. $\dfrac{4a^{-6}}{3b^{-7}}$. **76.** $\dfrac{5x^2y^{-1}}{x^{-4}y^2}$. **77.** $\dfrac{3^{-2}a^3}{5^{-3}b^{-2}}$. **78.** $\dfrac{2^{-3}x^{-1}}{5^{-1}a^{-4}}$. **79.** $\dfrac{3a^{-2}y^{-3}}{6^{-1}z^{-2}}$.

Write without a denominator by use of negative exponents.

80. $\dfrac{1}{y^3}$. **81.** $\dfrac{1}{z^5}$. **82.** $\dfrac{3}{x^2}$. **83.** $\dfrac{5}{y^4}$. **84.** $\dfrac{3x^2}{a^4}$.

85. $\dfrac{5z^2}{y^4}$. **86.** $\dfrac{8x^3}{y^3z^2}$. **87.** $\dfrac{4a^2}{3xy^3}$. **88.** $\dfrac{2a^6}{5z^2y^3}$. **89.** $\dfrac{8x^{\frac{1}{3}}}{y^{\frac{1}{2}}z}$.

90. $\dfrac{1}{(1.03)^5}$. **91.** $\dfrac{3}{(1.04)^{16}}$. **92.** $\dfrac{b^3}{(1.05)^8}$. **93.** $\dfrac{c}{x-5y}$.

Rewrite, expressing each fractional power as a radical and each radical as a fractional power.

94. $x^{\frac{1}{4}}$. **95.** $z^{\frac{1}{5}}$. **96.** $a^{\frac{5}{7}}$. **97.** $b^{\frac{4}{3}}$. **98.** $3a^{\frac{1}{2}}$.

99. $5c^{\frac{1}{3}}$. **100.** $ax^{\frac{1}{4}}$. **101.** $bx^{\frac{2}{3}}$. **102.** $\sqrt[5]{a^3}$. **103.** $\sqrt[7]{b^6}$.

104. $(5b)^{\frac{1}{3}}$. **105.** $(6c)^{\frac{1}{5}}$. **106.** $\sqrt[3]{y^{12}}$. **107.** $(2xy)^{\frac{1}{2}}$. **108.** $(4c^3)^2$.

109. $(7a^2)^{\frac{2}{3}}$. **110.** $(2x^3)^{\frac{3}{4}}$. **111.** $\sqrt[5]{a^8}$. **112.** $\sqrt[3]{b^{11}}$. **113.** $\sqrt[4]{5c^3}$.

114. $\sqrt[4]{c+d}$. **115.** $\sqrt{a^2-3b}$. **116.** $\sqrt[3]{(a+b)^2}$. **117.** $\sqrt[5]{(c-3d)^3}$.

118. $\sqrt{a^2+b^2}$. **119.** $\sqrt[3]{a^3-b^3}$. **120.** $\sqrt[3]{8+x^3}$. **121.** $\sqrt{4-x^2}$.

122. Compute $(-\frac{1}{2})^{-3}$; $(-\frac{1}{3})^{-2}$; $(-.04)^{-3}$.

116. Extension of the index laws

We have defined a^p if p is any rational number but we have proved the index laws only for positive integral exponents. A detailed discussion, which we shall omit (see Appendix, Note 2), shows that the formulas of Laws I to V of pages 142 and 143 apply if the exponents are any rational numbers.

Hereafter, unless otherwise specified, **to simplify** *an expression involving exponents* will mean to perform indicated operations as far as possible by use of Laws I to V, and to express the result *without zero or negative exponents.* Moreover, unless specifically requested, we shall not introduce radicals for powers having fractional exponents.

In operations involving exponents, it is frequently convenient to express the numerical coefficients of terms as products of powers of prime factors.

ILLUSTRATION 1. $(x^6)^{\frac{2}{3}} = x^{6\cdot\frac{2}{3}} = x^4$. $x^{\frac{1}{4}}x^{\frac{2}{3}} = x^{\frac{1}{4}+\frac{2}{3}} = x^{\frac{11}{12}}$.

$$\left(\frac{216x^5}{125x^{-1}}\right)^{\frac{2}{3}} = \left(\frac{8\cdot 27x^5x}{125}\right)^{\frac{2}{3}} = \left(\frac{2^3 3^3 x^6}{5^3}\right)^{\frac{2}{3}} = \frac{2^2 3^2 x^4}{5^2} = \frac{36x^4}{25}.$$

$$\left(-\frac{1}{125}\right)^{-\frac{2}{3}} = \left[\left(-\frac{1}{5}\right)^3\right]^{-\frac{2}{3}} = \left(-\frac{1}{5}\right)^{-2} = \frac{1}{(-\frac{1}{5})^2} = \frac{1}{\frac{1}{25}} = 25.$$

EXAMPLE 1. Simplify: $\dfrac{a^{-2}y^{-2}}{a^{-2}+y^{-2}}$.

FIRST SOLUTION. Change to positive exponents:

$$\frac{a^{-2}y^{-2}}{a^{-2}+y^{-2}} = \frac{\dfrac{1}{a^2y^2}}{\dfrac{1}{a^2}+\dfrac{1}{y^2}} = \frac{\dfrac{1}{a^2y^2}}{\dfrac{y^2+a^2}{a^2y^2}} = \frac{1}{a^2y^2}\cdot\frac{a^2y^2}{a^2+y^2} = \frac{1}{a^2+y^2}.$$

SECOND SOLUTION. Multiply both numerator and denominator by a^2y^2, to eliminate negative exponents:

$$\frac{a^{-2}y^{-2}}{a^{-2}+y^{-2}} = \frac{(a^{-2}y^{-2})(a^2y^2)}{(a^{-2}+y^{-2})(a^2y^2)}$$

$$= \frac{a^0y^0}{a^0y^2+a^2y^0} = \frac{1}{y^2+a^2}.$$

We may use the special products of Chapter 5 in multiplying or factoring polynomials involving negative or fractional exponents.

ILLUSTRATION 2. $(x^{-2} - y^{\frac{1}{3}})(x^{-2} + y^{\frac{1}{3}})$ (Type II, page 85)

$$= (x^{-2})^2 - (y^{\frac{1}{3}})^2 = x^{-4} - y^{\frac{2}{3}}.$$

ILLUSTRATION 3. $(x^{\frac{1}{4}} - 2y^{-1})^2$ (Type IV, page 85)

$$= (x^{\frac{1}{4}})^2 - 2(x^{\frac{1}{4}})(2y^{-1}) + (2y^{-1})^2 = x^{\frac{1}{2}} - 4x^{\frac{1}{4}}y^{-1} + 4y^{-2}.$$

ILLUSTRATION 4. $2x^{-2} + x^{-1} - 6 = (2x^{-1} - 3)(x^{-1} + 2)$.

EXERCISE 58

Simplify and, if no letters are involved, evaluate.

1. $x^{\frac{1}{4}}x^2$.
2. $x^{\frac{3}{4}}x^{\frac{1}{2}}$.
3. x^0x^4.
4. a^3a^0.
5. $(a^{\frac{3}{2}})^4$.
6. $(3^4)^{\frac{3}{2}}$.
7. $(2^6)^{\frac{2}{3}}$.
8. $(x^{\frac{3}{5}})^4$.
9. $(4x^2)^{\frac{3}{2}}$.
10. $(3x^{-1})^2$.
11. $(5a^{-2})^3$.
12. $(c^{-1}x^2)^3$.
13. $(5^3)^{\frac{2}{3}}$.
14. $(x^{\frac{2}{3}})^{\frac{1}{5}}$.
15. $(a^{\frac{2}{3}})^{\frac{1}{4}}$.
16. $(ax^{-1})^4$.
17. $(x^3)^{-1}$.
18. $(a^2)^{-2}$.
19. $(b^3)^{-2}$.
20. $(a^{-2})^{-3}$.
21. $(2x^{-2})^3$.
22. $(5a^{-3})^2$.
23. $(a^2x^3)^{-3}$.
24. $(b^{-2}x^2)^{-4}$.

25. $(3x^{-1}y^{3})^{2}$. **26.** $(5x^{-2}y^{-3})^{2}$. **27.** $(6x^{-1}y^{-3})^{-2}$. **28.** $(5a^{-3}b^{2})^{-2}$.

29. $32^{\frac{3}{5}}$. **30.** $27^{\frac{4}{3}}$. **31.** $125^{\frac{4}{3}}$. **32.** $216^{\frac{2}{3}}$.

33. $(4^{-3}x^{6})^{\frac{2}{3}}$. **34.** $(x^{-4}y^{2})^{\frac{5}{2}}$. **35.** $(27a^{-3}x^{6})^{\frac{2}{3}}$. **36.** $(25x^{-2})^{-\frac{3}{2}}$.

37. $\dfrac{x^{5}}{x^{\frac{7}{2}}}$. **38.** $\dfrac{a^{11}}{a^{\frac{10}{3}}}$. **39.** $\dfrac{a^{2}}{a^{\frac{10}{3}}}$. **40.** $\dfrac{x^{3}}{x^{\frac{11}{3}}}$. **41.** $\dfrac{x^{\frac{1}{3}}}{x^{\frac{5}{6}}}$.

42. $\dfrac{a^{\frac{3}{4}}}{a^{\frac{1}{2}}}$. **43.** $\dfrac{a^{\frac{1}{3}}x^{0}}{a^{2}x^{4}}$. **44.** $\dfrac{x^{3}y^{0}}{x^{\frac{1}{2}}y^{\frac{2}{3}}}$. **45.** $\dfrac{a^{3}b^{2}}{a^{\frac{5}{2}}b^{\frac{7}{2}}}$. **46.** $\dfrac{3x^{4}y}{5x^{\frac{1}{3}}y^{\frac{5}{3}}}$.

47. $\dfrac{3a^{\frac{1}{3}}y^{\frac{1}{4}}}{6a^{\frac{1}{4}}y^{\frac{1}{2}}}$. **48.** $\dfrac{a^{-1}b^{-2}}{a^{-2}b^{-1}}$. **49.** $\dfrac{a^{-3}b^{-2}}{a^{-4}b^{-5}}$. **50.** $\dfrac{x^{-3}y^{-5}}{x^{-1}y^{2}}$.

51. $\dfrac{a^{3}y^{-3}}{a^{-2}y^{-2}}$. **52.** $\left(\dfrac{2x^{2}}{5y^{\frac{1}{3}}}\right)^{2}$. **53.** $\left(\dfrac{2a^{3}x}{3a^{\frac{1}{2}}}\right)^{3}$. **54.** $\left(\dfrac{x^{\frac{1}{2}}y^{\frac{2}{3}}}{2a^{\frac{1}{2}}}\right)^{3}$.

55. $\left(\dfrac{a^{\frac{2}{3}}b^{\frac{1}{2}}}{3x^{\frac{1}{4}}}\right)^{4}$. **56.** $\left(\dfrac{3ax^{-1}}{a^{\frac{1}{2}}x^{-2}}\right)^{2}$. **57.** $\left(\dfrac{9x^{-2}}{4y^{4}}\right)^{\frac{1}{2}}$. **58.** $\left(\dfrac{a^{4}b^{-4}}{16x^{\frac{4}{3}}}\right)^{\frac{1}{4}}$.

59. $(27m^{6})^{\frac{2}{3}}$. **60.** $(32a^{5}b^{5})^{\frac{3}{5}}$. **61.** $(125x^{6})^{\frac{2}{3}}$. **62.** $(216x^{-6})^{\frac{2}{3}}$.

Simplify to a single fraction in lowest terms without negative exponents.

63. $a^{-1} + b^{-1}$. **64.** $3a^{-2} + b$. **65.** $a^{-3} - b^{-3}$. **66.** $5a^{-1} + b^{-3}$.

67. $\dfrac{a^{-1}b}{a^{-1} + b}$. **68.** $\dfrac{a^{-2} - b}{a^{-2} + b}$. **69.** $\dfrac{c^{-2}y^{-2}}{c^{-2} + y^{-2}}$. **70.** $\dfrac{3^{-2} - 4^{-2}}{3^{-2}4^{-2}}$.

71. $\dfrac{a^{-1} + b^{-1}}{a^{-2} - b^{-2}}$. **72.** $\dfrac{4^{-1} - a^{-1}}{4^{-2} - a^{-2}}$. **73.** $\dfrac{a^{-1} - b^{-1}}{a^{-3} - b^{-3}}$. **74.** $\dfrac{x^{-6} + y^{-6}}{x^{-2} + y^{-2}}$.

75. $(3ab)^{-1}$. **76.** $(c + 5d)^{-1}$. **77.** $(a^{-1} + b^{-2})^{-1}$. **78.** $(4a^{-3} - b)^{-1}$.

Expand and simplify, without eliminating negative exponents.

79. $(x^{-1} - y^{-1})(x^{-1} + y^{-1})$. **80.** $(3a - b^{-1})(3a + b^{-1})$.

81. $(4x - y^{\frac{1}{3}})(4x + y^{\frac{1}{3}})$. **82.** $(x^{\frac{2}{3}} - 4y)(x^{\frac{2}{3}} + 4y)$.

83. $(x^{\frac{1}{4}} + y^{\frac{1}{4}})(x^{\frac{1}{4}} - y^{\frac{1}{4}})$. **84.** $(2x^{\frac{1}{3}} - 3)(3x^{\frac{1}{3}} + 1)$.

85. $(5a^{\frac{1}{2}} - 2)(3a^{\frac{1}{2}} + 4)$. **86.** $(a^{\frac{1}{3}} - 2b^{\frac{1}{3}})(a^{\frac{1}{3}} + 3b^{\frac{1}{3}})$.

87. $(a^{-2} + b)^{2}$. **88.** $(x^{-1} + 3)^{2}$. **89.** $(a^{\frac{1}{3}} + b^{\frac{2}{3}})^{2}$.

90. $(2a^{\frac{1}{2}} + 3b^{\frac{1}{3}})^{2}$. **91.** $(a^{-1} + y^{2})^{2}$. **92.** $(x^{4} - 2y^{-1})^{2}$.

93. $(a^{-1} + b)^{3}$. **94.** $(2 + x^{-2})^{3}$. **95.** $(3 - y^{-1})^{3}$.

96. $(2x^{-2} - 3)(x^{-2} + 4)$. **97.** $(3 - 5x^{-2})(2 + 3x^{-2})$.

98. $(x^{3k}y^{5k})^{\frac{1}{k}}$. **99.** $(3x^{\frac{1}{4}})^n$. **100.** $(3^k a^k b^n)^h$. **101.** $(4a^{-k}b^n)^m$.

102. $\left(\dfrac{a^{6x}}{121x^{\frac{4}{3}}}\right)^{\frac{1}{2}}$. **103.** $\left(\dfrac{4x^{4k}}{49a^{6n}}\right)^{\frac{3}{2}}$. **104.** $\left(\dfrac{.125x^9}{8z^{3k}}\right)^{\frac{2}{3}}$.

105. $(a - b^{\frac{1}{2}})(a^2 + ab^{\frac{1}{2}} + b)$. **106.** $(c^{\frac{1}{3}} + d)(c^{\frac{2}{3}} - c^{\frac{1}{3}}d + d^2)$.

★*Factor into two factors involving fractional or negative exponents. When possible, factor as the difference of two squares or as a perfect square.*

ILLUSTRATION 1. $x - y = (x^{\frac{1}{2}})^2 - (y^{\frac{1}{2}})^2 = (x^{\frac{1}{2}} - y^{\frac{1}{2}})(x^{\frac{1}{2}} + y^{\frac{1}{2}})$.

Or, $x - y = (x^{\frac{1}{3}})^3 - (y^{\frac{1}{3}})^3 = (x^{\frac{1}{3}} - y^{\frac{1}{3}})(x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}})$.

107. $x^2 - y^{-2}$. **108.** $a^{-2} - b^4$. **109.** $9x^{-2} - b^{-4}$.

110. $y^4 - x^{\frac{2}{3}}$. **111.** $4x^{\frac{2}{3}} - y^{\frac{2}{5}}$. **112.** $9a^{\frac{2}{3}} - 4x$.

113. $9x^{\frac{2}{5}} - 25y$. **114.** $16a^{\frac{2}{3}} - 49y^{\frac{4}{5}}$. **115.** $4a - 9b$.

116. $x^2 - 2xy^{-1} + y^{-2}$. **117.** $z^2 - 6zx^{-1} + 9x^{-2}$.

118. $4a^{-2} - 4a^{-1}b^{-1} + b^{-2}$. **119.** $9a^{-2} - 6a^{-1}b^{-2} + b^{-4}$.

120. $4a^{\frac{2}{3}} - 20a^{\frac{1}{3}}b^{\frac{1}{3}} + 25b^{\frac{2}{3}}$. **121.** $36x^{\frac{4}{5}} + 12x^{\frac{2}{5}}y^{\frac{2}{5}} + y^{\frac{4}{5}}$.

122. $x^{-2} - 4x^{-1} - 5$. **123.** $3x^{-2} + x^{-1}y - 2y^2$.

124. $x - 9y^{\frac{2}{3}}$. **125.** $8a + 27b$. **126.** $x - 8y$. **127.** $216 - x^{\frac{3}{5}}$.

★*Find the quotient by use of factoring.*

128. $\dfrac{x^{-4} - y^{-4}}{x^{-2} + y^{-2}}$. **129.** $\dfrac{9x^{-2} - y^{-2}}{3x^{-1} - y^{-1}}$. **130.** $\dfrac{8x + y}{2x^{\frac{1}{3}} + y^{\frac{1}{3}}}$.

117. Simplification of a radicand

Although it is possible to express a radical as a power with a fractional exponent, in some operations it is convenient to retain the radical form. This remark applies in particular to the following operation.

SUMMARY. *To remove factors from the radicand in a radical of order n:*

1. *Separate the radicand into factors of which as many as possible are perfect* n*th powers.*

2. *Find the* n*th root of each perfect* n*th power and express the final result by use of the property* $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$.

ILLUSTRATION 1. $\sqrt{147} = \sqrt{49 \cdot 3} = \sqrt{49}\sqrt{3} = 7\sqrt{3};$

$\sqrt{147} = 7(1.732) = 12.124.$ (Table I)

$$\sqrt[5]{64a^{10}c^9} = \sqrt[5]{2^6a^{10}c^9} = \sqrt[5]{2^5a^{10}c^5 \cdot 2c^4} = \sqrt[5]{2^5a^{10}c^5}\sqrt[5]{2c^4} = 2a^2c\sqrt[5]{2c^4}.$$

ILLUSTRATION 2. $\sqrt[3]{3a + \dfrac{5}{x^3}} = \sqrt[3]{\dfrac{3ax^3 + 5}{x^3}} = \dfrac{\sqrt[3]{3ax^3 + 5}}{\sqrt[3]{x^3}} = \dfrac{\sqrt[3]{3ax^3 + 5}}{x}.$

Hereafter, unless otherwise specified, if a radicand involves fractions, reduce it to a *single* fraction. If a radical is of order n, simplify the radicand by removing from it any factor which is a perfect nth power. Also, in a radical of *odd* order, change the radicand to a form where all signs are "+" if possible.

ILLUSTRATION 3. $\sqrt[3]{-2} = \sqrt[3]{-1}\sqrt[3]{2} = -1 \cdot \sqrt[3]{2} = -\sqrt[3]{2}.$

$$\sqrt[3]{-a-2b} = \sqrt[3]{-(a+2b)} = \sqrt[3]{-1}\sqrt[3]{a+2b} = -\sqrt[3]{a+2b}.$$

In a sum, two or more terms involving the same radical as a factor may be combined by factoring.

ILLUSTRATION 4. $5\sqrt{3} + 2b\sqrt{3} = (5 + 2b)\sqrt{3}.$

$$\sqrt{20} + 2\sqrt{45} = \sqrt{4}\sqrt{5} + 2\sqrt{9}\sqrt{5} = 2\sqrt{5} + 6\sqrt{5} = 8\sqrt{5}.$$

$(\sqrt{3} - \sqrt{7} + 3\sqrt{5})$ cannot be simplified in form.

ILLUSTRATION 5. $7\sqrt{5} - 3\sqrt{5} + 6\sqrt{5} = \sqrt{5}(7 - 3 + 6) = 10\sqrt{5}.$

(Using $\sqrt{5} = 2.236$ from Table I) $= 10(2.236) = 22.36.$

EXERCISE 59

Simplify the radical and then compute by use of Table I.

1. $\sqrt{18}$. **2.** $\sqrt{75}$. **3.** $\sqrt{20}$. **4.** $\sqrt{24}$. **5.** $\sqrt{200}$.

6. $\sqrt{500}$. **7.** $\sqrt{27}$. **8.** $\sqrt{108}$. **9.** $\sqrt{72}$. **10.** $\sqrt{\frac{8}{100}}$.

11. $\sqrt{.45}$. **12.** $\sqrt[3]{24}$. **13.** $\sqrt[3]{16}$. **14.** $\sqrt[3]{54}$. **15.** $\sqrt[3]{108}$.

16. $\sqrt[3]{-3}$. **17.** $\sqrt[3]{-5}$. **18.** $\sqrt[3]{-16}$. **19.** $\sqrt[3]{-54}$ **20.** $\sqrt[3]{.024}$.

Simplify by removing perfect powers from the radicand.

21. $\sqrt{x^7}$. **22.** $\sqrt{a^9}$. **23.** $\sqrt[3]{y^5}$. **24.** $\sqrt[3]{a^8}$. **25.** $\sqrt[4]{x^{10}}$. **26.** $\sqrt[4]{z^{15}}$.

27. $\sqrt{9a^4}$. **28.** $\sqrt{8x^5}$. **29.** $\sqrt[3]{8a^4}$. **30.** $\sqrt[3]{27x^8}$.

31. $\sqrt[4]{16z^7}$. **32.** $\sqrt{20a^4x^6}$. **33.** $\sqrt{12a^3y^5}$. **34.** $\sqrt{50a^2y^4}$.

35. $\sqrt[3]{12a^3y^5}$. **36.** $\sqrt{75x^5}$. **37.** $\sqrt[3]{54y^5}$. **38.** $\sqrt[3]{375x^8}$.

39. $\sqrt[3]{16y^8z^4}$. **40.** $\sqrt[3]{-27a^8}$. **41.** $\sqrt[3]{-a^5y^6}$. **42.** $\sqrt[3]{-128x^6}$.

43. $\sqrt[4]{3x^4y^{10}}$. **44.** $\sqrt[4]{81a^5y^{11}}$. **45.** $\sqrt[4]{16c^3d^5}$. **46.** $\sqrt[4]{32a^4y^{16}}$.

47. $\sqrt[3]{-9a^8y^6}$. **48.** $\sqrt{.04a^9}$. **49.** $\sqrt{.25x^7}$. **50.** $\sqrt[3]{.008x^5}$.

51. $\sqrt{\dfrac{4x^5}{y^4}}$. **52.** $\sqrt{\dfrac{5x^6}{9y^4}}$. **53.** $\sqrt[3]{\dfrac{5x^5}{8y^6}}$. **54.** $\sqrt[3]{\dfrac{16a^4}{27x^9}}$.

55. $\sqrt[3]{\dfrac{216x^5}{125y^3}}$. **56.** $\sqrt[3]{-\dfrac{x^5}{8y^6}}$. **57.** $\sqrt[3]{-\dfrac{16}{a^3b^6}}$. **58.** $\sqrt[4]{\dfrac{4ac^5}{81x^4}}$.

59. $\sqrt{9 + 9y^2}$. **60.** $\sqrt{4 - 4a^2}$. **61.** $\sqrt{a^2 + 5a^2b}$.

62. $\sqrt{c^2d^2 + 4c^2d}$. **63.** $\sqrt[3]{16a^3 - 8a^4z^3}$. **64.** $\sqrt[3]{54x^3 + 27x^3y^3}$.

65. $\sqrt[3]{x^{3h}}$. **66.** $\sqrt[4]{5x^{4k}}$. **67.** $\sqrt[3]{16x^{3n}}$. **68.** $\sqrt[4]{16a^{8n}}$.

69. $\sqrt{\dfrac{d}{b^2} + \dfrac{2d}{ab} + \dfrac{d}{a^2}}$. **70.** $\sqrt{5 + \dfrac{10y}{x^2} + \dfrac{5y^2}{x^4}}$.

Simplify and then collect terms, exhibiting any common radical factor.

71. $5\sqrt{2} + 3\sqrt{2}$. **72.** $3\sqrt{3} - a\sqrt{3}$. **73.** $2\sqrt{18} + \sqrt{50}$.

74. $\sqrt{12} + \sqrt{75}$. **75.** $\sqrt{147} - \sqrt{48}$. **76.** $\sqrt[3]{24} + 2\sqrt[3]{81}$.

77. $a\sqrt{2} - 5b\sqrt{2}$. **78.** $\sqrt{9a} + \sqrt{25a}$. **79.** $\sqrt[3]{81x} - \sqrt[3]{3x^4}$.

80. $\sqrt[3]{8a^4} + \sqrt[3]{b^3a}$. **81.** $\sqrt[4]{48x^4y} - \sqrt[4]{48y}$. **82.** $\sqrt[3]{8x^3y} - \sqrt[3]{27a^3y}$.

118. Products and quotients of radicals

The product or the quotient of two radicals of the same order can be expressed as a single radical by use of the following properties of radicals.

$$\sqrt[n]{a}\,\sqrt[n]{b} = \sqrt[n]{ab}. \qquad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}.$$

Also, we recall that, by the definition of a root, $(\sqrt[n]{A})^n = A$.

ILLUSTRATION 1. $(2\sqrt{3})(5\sqrt{6}) = 10\sqrt{3}\sqrt{6}$

$$= 10\sqrt{18} = 30\sqrt{2}.$$

ILLUSTRATION 2. $(5\sqrt[3]{3})^3 = 5^3(\sqrt[3]{3})^3 = 125(3) = 375.$

ILLUSTRATION 3. $\dfrac{\sqrt{3}}{\sqrt{5}} = \sqrt{\dfrac{3}{5}}.$ $\dfrac{\sqrt[3]{ab}}{\sqrt[3]{b^5}} = \sqrt[3]{\dfrac{ab}{b^5}} = \sqrt[3]{\dfrac{a}{b^4}} = \dfrac{1}{b}\sqrt[3]{\dfrac{a}{b}}.$

ILLUSTRATION 4. $\sqrt[3]{2a}\sqrt[3]{20a^2b^3} = \sqrt[3]{40a^3b^3} = \sqrt[3]{8a^3b^3}\sqrt[3]{5} = 2ab\sqrt[3]{5}.$

EXAMPLE 1. Multiply $(2\sqrt{3} + 3\sqrt{2x})(3\sqrt{3} - \sqrt{2x}).$

SOLUTION. The product is

$$\begin{aligned} &6(\sqrt{3})^2 - 2\sqrt{3}\sqrt{2x} + 9\sqrt{3}\sqrt{2x} - 3(\sqrt{2x})^2 \\ &= 18 - 2\sqrt{6x} + 9\sqrt{6x} - 3(2x) = 18 - 6x + 7\sqrt{6x}. \end{aligned}$$

Comment. The student may prefer an expanded solution:

$$\begin{array}{ll} 2\sqrt{3} + 3\sqrt{2x} & \\ \underline{3\sqrt{3} - \sqrt{2x}} & (\textit{multiply}) \\ 6(\sqrt{3})^2 + 9\sqrt{3}\sqrt{2x} & \\ \underline{\qquad\quad - 2\sqrt{3}\sqrt{2x} - 3(\sqrt{2x})^2} & \\ 18 \quad + 7\sqrt{3}\sqrt{2x} - 3(2x) = 18 - 6x + 7\sqrt{6x}. & \end{array}$$

If we remove a positive factor P multiplying a radical $\sqrt[n]{A}$ we must *multiply by* P^n *under the radical sign* because

$$P\sqrt[n]{A} = \sqrt[n]{P^n}\sqrt[n]{A} = \sqrt[n]{P^nA}.$$

ILLUSTRATION 5. $3\sqrt{b} = \sqrt{9}\sqrt{b} = \sqrt{9b}.$

EXERCISE 60

Express by means of a single radical, and then compute by use of Table I *if necessary.*

1. $\sqrt{5}\sqrt{3}.$
2. $\sqrt{2}\sqrt{3}.$
3. $\sqrt{5}\sqrt{10}.$
4. $\sqrt{3}\sqrt{15}.$
5. $\sqrt{6}\sqrt{12}.$
6. $(\sqrt[3]{2})^3.$
7. $(2\sqrt[3]{5})^3.$
8. $(3\sqrt{5})^2.$
9. $3\sqrt{3}(2\sqrt{6}).$
10. $5\sqrt{6}(2\sqrt{21}).$
11. $\sqrt{30}\sqrt{35}.$
12. $(2\sqrt{3}\sqrt{5})^2.$
13. $(3\sqrt[3]{2})^3.$
14. $\sqrt[3]{5}\sqrt[3]{50}.$
15. $3\sqrt[3]{36}(\sqrt[3]{45}).$
16. $\sqrt[3]{-4}\sqrt[3]{18}.$
17. $\sqrt[3]{-2}\sqrt[3]{12}.$
18. $\dfrac{\sqrt{15}}{\sqrt{5}}.$
19. $\dfrac{\sqrt{14}}{\sqrt{2}}.$
20. $\dfrac{\sqrt[3]{10}}{\sqrt[3]{5}}.$
21. $\dfrac{\sqrt[3]{99}}{\sqrt[3]{11}}.$
22. $\dfrac{\sqrt[3]{-81}}{\sqrt[3]{3}}.$

Express as a single radical and then simplify.

23. $\dfrac{\sqrt{10x}}{\sqrt{2x}}.$
24. $\dfrac{\sqrt{2d}}{\sqrt{4c}}.$
25. $\dfrac{\sqrt{3x^3}}{\sqrt{12x}}.$
26. $\dfrac{\sqrt[3]{5h^4}}{\sqrt[3]{40h}}.$
27. $\dfrac{\sqrt[3]{6a^5}}{\sqrt[3]{2a^8}}.$

Multiply, simplify, and collect terms.

28. $\sqrt{3a}\sqrt{15a}.$
29. $3\sqrt{x}\sqrt{5x^3}.$
30. $\sqrt{x}\sqrt{2x}\sqrt{6x^3}.$

31. $\sqrt{3}\sqrt{x}\sqrt{6x^2}$. **32.** $\sqrt[3]{5x}\sqrt[3]{25x^5y^4}$. **33.** $\sqrt[3]{9a^2}\sqrt[3]{6a^3b^4}$.

34. $(3\sqrt{2x})^2$. **35.** $(5\sqrt{3a^2})^3$. **36.** $(2\sqrt[3]{4a})^4$. **37.** $(3\sqrt[3]{2x})^3$.

38. $(5\sqrt{x+y})^2$. **39.** $(b\sqrt{x^2+1})^2$. **40.** $(-3\sqrt[3]{5b^2})^3$.

41. $(2+3\sqrt{5})(3-\sqrt{5})$. **42.** $(2\sqrt{2}-3)(5\sqrt{2}+2)$.

43. $(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})$. **44.** $(2\sqrt{3}-\sqrt{5})(3\sqrt{3}+\sqrt{5})$.

45. $(3\sqrt{2}+\sqrt{3})(\sqrt{2}+4\sqrt{3})$. **46.** $(\sqrt{6}-\sqrt{5})(\sqrt{6}+\sqrt{5})$.

47. $(2\sqrt{3}-\sqrt{7})(2\sqrt{3}+\sqrt{7})$. **48.** $(5\sqrt{2}-3\sqrt{3})(5\sqrt{2}+3\sqrt{3})$.

49. $(2\sqrt{3}+\sqrt{5})(\sqrt{2}+4)$. **50.** $(3\sqrt{2}+\sqrt{3})(\sqrt{2}+\sqrt{6})$.

51. $(\sqrt{2}+5)^2$. **52.** $(3+5\sqrt{2})^2$. **53.** $(\sqrt{2}-2\sqrt{3})^2$.

54. $(\sqrt{x}-2\sqrt{z})(\sqrt{x}+2\sqrt{z})$. **55.** $(\sqrt{a}+3\sqrt{by})(\sqrt{a}-3\sqrt{by})$.

56. $(2\sqrt[3]{x}-3)(5\sqrt[3]{x^2}+2)$. **57.** $(3\sqrt{x}+5\sqrt{y})(2\sqrt{x}-3\sqrt{y})$.

58. $(\sqrt{x}-5\sqrt{y})^2$. **59.** $(\sqrt{a}+b\sqrt{x})^2$. **60.** $(a\sqrt{y}-c\sqrt{z})^2$.

61. $\sqrt[5]{x^2y^4z^5}\sqrt[5]{-x^3yz^4}$. **62.** $\sqrt[3]{-3a^2b}\sqrt[3]{18a^4b^5}$.

Replace the coefficient by an equivalent factor under the radical sign.

63. $3\sqrt{2a}$. **64.** $3\sqrt{6x}$. **65.** $a\sqrt{bx}$. **66.** $c\sqrt{dx^2}$.

67. $3\sqrt[3]{b^2}$. **68.** $2\sqrt[3]{a}$. **69.** $2\sqrt[4]{3b}$. **70.** $3\sqrt[4]{ax}$.

119. Rationalization of denominators

To rationalize a denominator in a radical of order n, after the radicand has been expressed as a simple fraction, *multiply both numerator and denominator of the radicand by the simplest expression which will make the denominator a perfect* nth *power*. In particular, if the radical is a *square* root, we make the denominator a perfect *square;* if a *cube* root, we make the denominator a perfect *cube.*

ILLUSTRATION 1. $$\sqrt{\frac{3}{7}}=\sqrt{\frac{3\cdot 7}{7^2}}=\frac{\sqrt{21}}{7}=\frac{4.583}{7}=.655. \qquad \text{(Table I)}$$

Notice the inconvenience of the following attempt at computation of $\sqrt{\frac{3}{7}}$ without rationalization of the denominator; a long division is required.

$$\sqrt{\frac{3}{7}}=\frac{\sqrt{3}}{\sqrt{7}}=\frac{1.732}{2.646}=\text{etc.} \qquad \text{(Table I)}$$

ILLUSTRATION 2. $$\sqrt{\frac{3}{8}}=\sqrt{\frac{3\cdot 2}{8\cdot 2}}=\frac{\sqrt{6}}{\sqrt{16}}=\frac{\sqrt{6}}{4}.$$

ILLUSTRATION 3. $\sqrt[3]{\dfrac{3}{4}} = \sqrt[3]{\dfrac{3\cdot 2}{4\cdot 2}} = \dfrac{\sqrt[3]{6}}{2}.$

$$\sqrt[3]{\frac{64x^{-4}}{9}} = \sqrt[3]{\frac{64}{9x^4}} = \sqrt[3]{\frac{4^3\cdot 3x^2}{9x^4\cdot 3x^2}} = \frac{\sqrt[3]{4^3\cdot 3x^2}}{\sqrt[3]{3^3x^6}} = \frac{4\sqrt[3]{3x^2}}{3x^2}.$$

$$\sqrt{\frac{1}{a^2}+\frac{1}{b^3}} = \sqrt{\frac{b^3+a^2}{a^2b^3}} = \sqrt{\frac{b(b^3+a^2)}{a^2b^3b}} = \frac{\sqrt{b(b^3+a^2)}}{ab^2}.$$

EXERCISE 61

Compute by use of Table I after rationalizing the denominator.

1. $\sqrt{\frac{1}{2}}$. 2. $\sqrt{\frac{1}{3}}$. 3. $\sqrt{\frac{2}{5}}$. 4. $\sqrt{\frac{2}{7}}$. 5. $\sqrt{\frac{5}{8}}$.

6. $\sqrt{\frac{2}{27}}$. 7. $\sqrt[3]{\frac{1}{4}}$. 8. $\sqrt[3]{\frac{1}{9}}$. 9. $\sqrt[3]{\frac{5}{16}}$. 10. $\sqrt[3]{\frac{4}{25}}$.

11. $\sqrt[3]{-\frac{7}{1000}}$. 12. $\sqrt[3]{-\frac{1}{25}}$. 13. $\sqrt{\frac{3}{1000}}$. 14. $\sqrt[3]{\frac{1}{10}}$.

15. $\sqrt[3]{-.03}$. 16. $\sqrt{.007}$. 17. $\sqrt{.012}$. 18. $\sqrt[3]{-.128}$.

Eliminate any negative exponents, rationalize the denominators, and collect terms involving a common radical factor.

19. $\sqrt{\dfrac{x}{3}}$. 20. $\sqrt{\dfrac{3a}{5}}$. 21. $\sqrt{\dfrac{2}{a}}$. 22. $\sqrt{\dfrac{3a}{b}}$. 23. $\sqrt{\dfrac{1}{3x^2}}$.

24. $\sqrt{\dfrac{3a}{2x^3}}$. 25. $\sqrt[3]{\dfrac{c}{4}}$. 26. $\sqrt[3]{\dfrac{5a}{9}}$. 27. $\sqrt[3]{\dfrac{a}{2b}}$. 28. $\sqrt[3]{\dfrac{c}{3d^2}}$.

29. $\sqrt{\dfrac{5a}{2b^2}}$. 30. $\sqrt{\dfrac{3x^3}{8y^2}}$. 31. $\sqrt{\dfrac{1}{5zw^2}}$. 32. $\sqrt{\dfrac{x^3y^3}{30}}$. 33. $\sqrt[3]{\dfrac{b}{9a^4}}$.

34. $\sqrt[3]{\dfrac{1}{5a^5}}$. 35. $\sqrt[3]{\dfrac{d}{8c^2}}$. 36. $\sqrt{\dfrac{1}{7x^3}}$. 37. $\sqrt[3]{\dfrac{8n^3}{3x^5}}$. 38. $\sqrt[4]{\dfrac{a}{8}}$.

39. $\sqrt[4]{\dfrac{cx^2}{27}}$. 40. $\sqrt[4]{\dfrac{9a}{x^3}}$. 41. $\sqrt[4]{\dfrac{3x^2y}{a^2x^5}}$. 42. $\sqrt[4]{\dfrac{2ab^3}{c^3d^6}}$.

43. $\sqrt[3]{\dfrac{-1}{5a^3b^8}}$. 44. $\sqrt[3]{\dfrac{-4ab}{9xz^7}}$. 45. $\sqrt{\dfrac{4h}{a+x}}$. 46. $\sqrt{\dfrac{a-b}{a+b}}$.

47. $\sqrt{x^{-5}}$. 48. $\sqrt{b^{-3}}$. 49. $\sqrt[3]{x^{-2}}$. 50. $\sqrt[3]{a^{-7}b^{-2}}$. 51. $\sqrt{\frac{1}{5}x^{-3}}$.

52. $\sqrt{\dfrac{a}{3}+\dfrac{5}{x}}$. 53. $\sqrt{\dfrac{1}{b}-\dfrac{9}{5a^2}}$. 54. $\sqrt{\dfrac{3}{c}-\dfrac{c}{2b}}$. 55. $\sqrt{16+\dfrac{3}{7x^3}}$.

56. $\sqrt{\frac{1}{2}+x^{-3}}$. 57. $\sqrt{c^{-3}+d^{-2}}$. 58. $\sqrt[3]{\frac{1}{7}+x^{-5}}$. 59. $\sqrt[3]{a+\frac{1}{3}y^{-2}}$

60. $\sqrt[3]{-\frac{2}{9}}+2\sqrt[3]{-48}$. 61. $5\sqrt{\frac{1}{5}}+\sqrt{45}$. 62. $10\sqrt{\frac{2}{5}}-\sqrt{\frac{1}{10}}$.

63. $4\sqrt{\frac{1}{2}xy}-\sqrt{8xy}$. 64. $\sqrt[3]{\frac{1}{4}x}+\sqrt[3]{2x^4}$. 65. $\sqrt[3]{a^{-1}}+\sqrt[3]{27a^4}$

120. Additional devices for rationalizing denominators

The method of the following illustration is frequently equivalent to the procedure of the preceding section.

ILLUSTRATION 1. The denominator below is multiplied by $\sqrt[3]{2}$ in order to make the new radicand a perfect cube:

$$\frac{\sqrt[3]{7}}{\sqrt[3]{4}} = \frac{\sqrt[3]{7}}{\sqrt[3]{4}} \cdot \frac{\sqrt[3]{2}}{\sqrt[3]{2}} = \frac{\sqrt[3]{14}}{\sqrt[3]{8}} = \frac{\sqrt[3]{14}}{2}.$$

ILLUSTRATION 2. $$\frac{\sqrt{3}}{\sqrt{5}} = \frac{\sqrt{3}}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{15}}{5}.$$

If a denominator has the form $a\sqrt{b} - c\sqrt{d}$, we can rationalize it by multiplying by $a\sqrt{b} + c\sqrt{d}$ because

$$(a\sqrt{b} - c\sqrt{d})(a\sqrt{b} + c\sqrt{d}) = (a\sqrt{b})^2 - (c\sqrt{d})^2 = a^2b - c^2d.$$

ILLUSTRATION 3. $$\frac{3\sqrt{2} - \sqrt{3}}{2\sqrt{2} - \sqrt{3}} = \frac{3\sqrt{2} - \sqrt{3}}{2\sqrt{2} - \sqrt{3}} \cdot \frac{2\sqrt{2} + \sqrt{3}}{2\sqrt{2} + \sqrt{3}}$$

$$= \frac{6(\sqrt{2})^2 + (3 - 2)\sqrt{6} - (\sqrt{3})^2}{(2\sqrt{2})^2 - (\sqrt{3})^2} = \frac{9 + \sqrt{6}}{8 - 3} = \frac{9 + 2.449}{5} = 2.290.$$

In finding the quotient of two radicals, it may be desirable to write the expression as a single radical before rationalizing.

ILLUSTRATION 4. $$\frac{\sqrt[3]{3x^3y^7}}{\sqrt[3]{81x^5y^5}} = \sqrt[3]{\frac{3x^3y^7}{81x^5y^5}} = \sqrt[3]{\frac{y^2}{27x^2}} = \frac{\sqrt[3]{xy^2}}{3x}.$$

EXERCISE 62

Rationalize the denominator and, if no letters are involved, compute by use of Table I. *Collect terms in any polynomial.*

1. $\frac{1}{\sqrt{3}}$.
2. $\frac{1}{\sqrt{5}}$.
3. $\frac{6}{\sqrt{5}}$.
4. $\frac{3}{\sqrt{7}}$.
5. $\frac{2}{\sqrt{3}}$.
6. $\frac{3}{\sqrt{11}}$.
7. $\frac{\sqrt{5}}{\sqrt{3}}$.
8. $\frac{\sqrt{7}}{\sqrt{2}}$.
9. $\frac{2\sqrt{3}}{\sqrt{5}}$.
10. $\frac{3\sqrt{7}}{2\sqrt{5}}$.
11. $\frac{3}{\sqrt[3]{4}}$.
12. $\frac{1}{\sqrt[3]{100}}$.
13. $\frac{2 - \sqrt{3}}{3 + \sqrt{3}}$.
14. $\frac{2\sqrt{3} + 5}{\sqrt{3} - 4}$.
15. $\frac{1}{3 + 2\sqrt{2}}$.
16. $\frac{3}{\sqrt{3} - 3\sqrt{2}}$.
17. $\frac{\sqrt{2} - \sqrt{3}}{2\sqrt{2} + \sqrt{3}}$.
18. $\frac{\sqrt{2} - 2\sqrt{3}}{\sqrt{2} + \sqrt{3}}$.

19. $\dfrac{\sqrt{3}-\sqrt{5}}{2\sqrt{3}-\sqrt{5}}.$ **20.** $\dfrac{\sqrt{2}+3\sqrt{3}}{2\sqrt{3}+3\sqrt{2}}.$ **21.** $\dfrac{\sqrt{2}+\sqrt{5}}{3\sqrt{5}-\sqrt{2}}.$

22. $\dfrac{\sqrt{3}-\sqrt{7}}{2\sqrt{3}-3\sqrt{7}}.$ **23.** $\dfrac{\sqrt{7}-\sqrt{6}}{2\sqrt{6}+\sqrt{7}}.$ **24.** $\dfrac{2\sqrt{7}-\sqrt{3}}{2\sqrt{7}+\sqrt{3}}.$

25. $(3-\sqrt{2}) \div (2+\sqrt{2}).$ **26.** $(\sqrt{3}+\sqrt{11}) \div (1-\sqrt{11}).$

27. $\dfrac{\sqrt[3]{-3}}{\sqrt[3]{36}}.$ **28.** $\dfrac{\sqrt[3]{2}}{\sqrt[3]{49}}.$ **29.** $\dfrac{1}{\sqrt[3]{-10}}.$ **30.** $\dfrac{\sqrt[3]{3}}{\sqrt[3]{-49}}.$ **31.** $\dfrac{\sqrt[3]{-2}}{\sqrt[3]{-50}}.$

32. $\dfrac{\sqrt{10x}}{\sqrt{2x^2}}.$ **33.** $\dfrac{\sqrt{20x}}{\sqrt{60x^3}}.$ **34.** $\dfrac{\sqrt[3]{60a^5}}{\sqrt[3]{20a^7}}.$ **35.** $\dfrac{\sqrt[3]{8tz^2}}{\sqrt[3]{-32t^2}}.$

36. $\dfrac{\sqrt{\frac{1}{2}ab^2}}{\sqrt{5a^2b^3}}.$ **37.** $\dfrac{1}{\sqrt[3]{-2a^2b}}.$ **38.** $\dfrac{\sqrt[5]{39a^2}}{\sqrt[5]{13x^6}}.$ **39.** $\dfrac{\sqrt[3]{\frac{1}{4}bc}}{\sqrt[3]{\frac{1}{2}b^2c^2}}.$

40. $\dfrac{1}{\sqrt[5]{-16a^3b^2}}.$ **41.** $\dfrac{\sqrt[6]{x^2y}}{\sqrt[6]{27xy^4}}.$ **42.** $\dfrac{3\sqrt{c}+a}{2\sqrt{c}-\sqrt{2a}}.$

43. $\dfrac{\sqrt{x-2}}{3+2\sqrt{x+2}}.$ **44.** $\dfrac{\sqrt{3a}-1}{\sqrt{a+b}-\sqrt{2a}}.$ **45.** $\dfrac{3+2\sqrt[3]{4}}{\sqrt[3]{2}}.$

46. $\dfrac{3}{\sqrt{3}-\sqrt{2}+\sqrt{5}}.$ **47.** $\dfrac{2}{\sqrt{3}-\sqrt{6}+\sqrt{5}}.$ **48.** $\dfrac{\sqrt{a}}{\sqrt{a}-\sqrt{c}}.$

49. $\sqrt[n]{\dfrac{3}{x^2}}.$ **50.** $\sqrt[k]{\dfrac{a}{2b^3}}.$ **51.** $\sqrt[h]{\dfrac{b^hx}{a^3x^{2h}}}.$ **52.** $\sqrt[m]{\dfrac{2x^{3m}}{5a^2y^3}}.$

121. Changing from fractional exponents to radicals

To change a product of powers involving fractional exponents to radical form, first change the fractional parts of the exponents to fractions with their lowest common denominator.

ILLUSTRATION 1. $5y^{\frac{2}{3}}z^{\frac{5}{2}} = 5z^2y^{\frac{2}{3}}z^{\frac{1}{2}} = 5z^2y^{\frac{4}{6}}z^{\frac{3}{6}} = 5z^2(y^4z^3)^{\frac{1}{6}} = 5z^2\sqrt[6]{y^4z^3}.$

122. Radical operations performed by using fractional exponents

In this section the results will be desired in *radical* form.

I. *To find a power or a root of a radical:*

1. *Express each radical operation by use of a fractional exponent applied to the radicand.*

2. *Simplify the indicated power of the radicand and express the result as a radical.*

ILLUSTRATION 1. $\sqrt[3]{\sqrt[4]{3xy}} = [(3xy)^{\frac{1}{4}}]^{\frac{1}{3}} = (3xy)^{\frac{1}{12}} = \sqrt[12]{3xy}.$

ILLUSTRATION 2. $(2\sqrt[3]{5x})^2 = 4[(5x)^{\frac{1}{3}}]^2 = 4(5x)^{\frac{2}{3}} = 4\sqrt[3]{(5x)^2} = 4\sqrt[3]{25x^2}.$

We recognize that, in simple cases, it may be unnecessary to introduce fractional exponents in an operation of Type I. Also, with experience, one observes simple rules such as

"*the* m*th root of the* n*th root of A is the* mn*th root of A.*" (1)

ILLUSTRATION 3. $(\sqrt[3]{3})^3 = 3.\quad \sqrt[3]{\sqrt{a}} = \sqrt[6]{a}.$

$$(\sqrt[5]{2})^2 = \sqrt[5]{2}\sqrt[5]{2} = \sqrt[5]{4}.$$

$$(\sqrt{3x})^5 = [(3x)^{\frac{1}{2}}]^5 = (3x)^{\frac{5}{2}} = (3x)^2(3x)^{\frac{1}{2}} = 9x^2\sqrt{3x}.$$

II. *To find the product or the quotient of two radicals of different orders:*

1. *Express each radical as a fractional power of its radicand, and change the resulting fractional exponents to their* LCD.

2. *Rewrite in radical form and combine into a single radical.*

ILLUSTRATION 4. $\sqrt{3}\sqrt[5]{2} = 3^{\frac{1}{2}}2^{\frac{1}{5}} = 3^{\frac{5}{10}}2^{\frac{2}{10}} = \sqrt[10]{3^5}\sqrt[10]{2^2} = \sqrt[10]{3^52^2}.$

ILLUSTRATION 5. $\dfrac{\sqrt[3]{4b^2x}}{\sqrt{3ab}} = \dfrac{(4b^2x)^{\frac{1}{3}}}{(3ab)^{\frac{1}{2}}} = \dfrac{(4b^2x)^{\frac{2}{6}}}{(3ab)^{\frac{3}{6}}} = \sqrt[6]{\dfrac{(4b^2x)^2}{(3ab)^3}}$

$$= \sqrt[6]{\frac{16bx^2}{27a^3}} = \frac{\sqrt[6]{27\cdot 16a^3bx^2}}{3a}.$$

III. *To reduce the order of a radical, when possible:*

1. *Change to fractional exponents in lowest possible terms with a common denominator.*

2. *Rewrite finally in radical form.*

ILLUSTRATION 6. $\sqrt[6]{625} = \sqrt[6]{5^4} = 5^{\frac{4}{6}} = 5^{\frac{2}{3}} = \sqrt[3]{5^2} = \sqrt[3]{25}.$

$$\sqrt[12]{x^2y^8} = (x^2y^8)^{\frac{1}{12}} = x^{\frac{2}{12}}y^{\frac{8}{12}} = x^{\frac{1}{6}}y^{\frac{4}{6}} = \sqrt[6]{xy^4}.$$

In reducing the order of a radical, it is convenient to commence by expressing the radicand as a power of some expression.

ILLUSTRATION 7. $\sqrt[8]{16x^2} = \sqrt[8]{(4x)^2} = (4x)^{\frac{2}{8}} = (4x)^{\frac{1}{4}} = \sqrt[4]{4x}.$

123. Simplest radical form

As far as problems in this text are concerned, we agree that an expression is in its *simplest radical form* if all possible operations of the following varieties have been performed, with any negative exponents eliminated.

SUMMARY. *To reduce a radical expression to simplest form:*

1. *Express any power or root of a radical, or product of radicals, as a single radical.*
2. *Reduce each radicand to a simple fraction in lowest terms.*
3. *Rationalize all denominators.*
4. *Remove from each radicand all factors which are perfect* nth *powers, where n is the order of the radical.*
5. *Reduce each radical to the lowest possible order.*
6. *Combine any terms with a common radical factor.*

It must not be inferred that the preceding operations need be performed in the specified order.

To simplify a radical expression will mean to reduce it to *simplest radical form.*

ILLUSTRATION 1. To simplify the following radical we rationalize the denominator, and finally notice that the order of the radical can be reduced.

$$\sqrt[6]{\frac{a^2}{16c^{10}}} = \sqrt[6]{\frac{a^2 \cdot 4c^2}{16 \cdot 4c^{12}}} = \frac{\sqrt[6]{4a^2c^2}}{2c^2} = \frac{\sqrt[6]{(2ac)^2}}{2c^2} = \frac{\sqrt[3]{2ac}}{2c^2}.$$

EXERCISE 63

Change to simplest radical form.

1. $a^{\frac{1}{3}}b^{\frac{1}{2}}$. **2.** $x^{\frac{2}{3}}y^{\frac{1}{2}}$. **3.** $5a^{\frac{3}{4}}$. **4.** $2x^{\frac{3}{7}}$. **5.** $2a^{\frac{1}{4}}b^{\frac{2}{3}}$.

6. $a^{\frac{2}{3}}b^{\frac{1}{6}}$. **7.** $a^{\frac{2}{3}}b^{\frac{3}{2}}$. **8.** $x^{\frac{1}{6}}y^{\frac{2}{3}}$. **9.** $x^{\frac{1}{2}}y^{\frac{3}{4}}$. **10.** $ad^{\frac{1}{3}}y^{\frac{1}{4}}$.

Reduce to a radical of lower order.

11. $\sqrt[12]{y^4}$. **12.** $\sqrt[8]{w^2}$. **13.** $\sqrt[4]{x^2}$. **14.** $\sqrt[6]{y^2}$. **15.** $\sqrt[12]{y^8}$.

16. $\sqrt[6]{z^3}$. **17.** $\sqrt[9]{z^3}$. **18.** $\sqrt[10]{x^4}$. **19.** $\sqrt[4]{9}$. **20.** $\sqrt[4]{25}$.

21. $\sqrt[6]{27}$. **22.** $\sqrt[6]{125}$. **23.** $\sqrt[8]{36}$. **24.** $\sqrt[8]{49}$. **25.** $\sqrt[10]{32}$.

26. $\sqrt[8]{81}$. **27.** $\sqrt[4]{9a^2}$. **28.** $\sqrt[4]{4z^2}$. **29.** $\sqrt[6]{8a^3}$. **30.** $\sqrt[6]{27b^3}$.

Change to simplest radical form.

31. $(\sqrt{b})^3$. **32.** $(\sqrt{c})^4$. **33.** $(\sqrt{x})^5$. **34.** $(\sqrt[3]{z})^4$. **35.** $(\sqrt[3]{x})^2$.

36. $(\sqrt[4]{a})^5$. **37.** $(\sqrt{3})^4$. **38.** $(\sqrt{2})^3$. **39.** $(\sqrt[3]{5})^4$. **40.** $(\sqrt[3]{2})^5$.

41. $(\sqrt{5})^5$. **42.** $(\sqrt{6})^4$. **43.** $(\sqrt{3a})^4$. **44.** $(\sqrt{2x})^5$. **45.** $(\sqrt{3a^5})^3$.

46. $(\sqrt[3]{2x})^4$. **47.** $(2\sqrt[3]{3})^4$. **48.** $(\sqrt[3]{ax^2})^5$. **49.** $\sqrt[3]{\sqrt{x}}$. **50.** $\sqrt[5]{\sqrt{z}}$.

51. $\sqrt{\sqrt[3]{y}}$. **52.** $\sqrt[3]{\sqrt[5]{5}}$. **53.** $\sqrt{\sqrt{3}}$. **54.** $\sqrt[3]{\sqrt{5}}$. **55.** $\sqrt[3]{\sqrt[3]{a}}$.

56. $\sqrt[4]{\sqrt[4]{x}}$. **57.** $\sqrt[3]{\sqrt[4]{y^3}}$. **58.** $\sqrt{\sqrt[3]{x^2}}$. **59.** $\sqrt[3]{\sqrt[4]{a^3}}$. **60.** $\sqrt{\sqrt[5]{a^2}}$.

61. $\sqrt{a}\sqrt[3]{a}$. **62.** $\sqrt{x}\sqrt[4]{x}$. **63.** $\sqrt[3]{y}\sqrt[4]{y}$. **64.** $\sqrt[5]{x}\sqrt[3]{x}$.

65. $\sqrt{3}\sqrt[4]{3}$. **66.** $\sqrt[3]{2}\sqrt[4]{2}$. **67.** $\sqrt[4]{a^2}\sqrt[3]{a^2}$. **68.** $\sqrt[3]{a^2}\sqrt[4]{a^3}$.

69. $\sqrt[3]{a} \div \sqrt{a}$. **70.** $\sqrt[3]{a} \div \sqrt[4]{a}$. **71.** $\sqrt[4]{b} \div \sqrt{b}$.

72. $\sqrt{2} \div \sqrt[3]{2}$. **73.** $\sqrt[3]{5} \div \sqrt[4]{5}$. **74.** $\sqrt[3]{3} \div \sqrt[6]{9}$.

75. $\sqrt[6]{8} \div \sqrt{2}$. **76.** $\sqrt[4]{3} \div \sqrt[8]{81}$. **77.** $\sqrt{6} \div \sqrt[3]{16}$.

78. $\sqrt[3]{a^2b} \div \sqrt{ab}$. **79.** $\sqrt[4]{cd^3} \div \sqrt{cd}$. **80.** $3\sqrt{y} \div \sqrt[3]{x^2y^2}$.

81. $\sqrt[4]{\frac{4}{25}}$. **82.** $\sqrt[6]{\frac{27}{8}}$. **83.** $\sqrt[3]{-\frac{1}{2}x^7}$. **84.** $\sqrt[3]{-\frac{1}{4}x^5}$. **85.** $\sqrt[4]{\frac{1}{9}x^{-3}}$.

86. $\sqrt{x^{-5}}$. **87.** $\sqrt[3]{16y^{-4}}$. **88.** $\sqrt[3]{216}$. **89.** $\sqrt[3]{\sqrt[4]{8a^3}}$.

90. $\sqrt[3]{27\sqrt{a^3}}$. **91.** $(\sqrt[4]{ax^2})^5$. **92.** $\sqrt{9\sqrt[3]{4x^2}}$. **93.** $\sqrt[3]{\sqrt{9b^9}}$.

94. $\sqrt[3]{x^2\sqrt{8}}$. **95.** $\sqrt{3}\sqrt[3]{3}\sqrt[4]{3}$. **96.** $\sqrt[3]{x^{-2} + y^{-2}}$. **97.** $\sqrt{x^{-4} + y^{-2}}$.

98. $\sqrt[6]{\frac{a^8}{b^2}}$. **99.** $\sqrt[6]{\frac{a^{10}}{4b^2}}$. **100.** $\sqrt[4]{\frac{64b^6}{9a^2}}$. **101.** $\sqrt[4]{\frac{9x^2y^6}{4b^4}}$. **102.** $\sqrt[6]{\frac{b^{10}a^4}{9x^2}}$.

103. $(\sqrt[3]{ax^2})^7$. **104.** $(2\sqrt[3]{5})^4$. **105.** $\sqrt[3]{\sqrt{\sqrt{a}}}$. **106.** $\sqrt{\sqrt{\sqrt{2}}}$.

107. $\sqrt{\sqrt[3]{9y^2}}$. **108.** $\sqrt{\sqrt[3]{4x^2}}$. **109.** $(a\sqrt[3]{2})^5$. **110.** $(b\sqrt[4]{3})^6$.

111. $(c\sqrt[5]{4})^3$. **112.** $\sqrt[6]{2a^4}\sqrt[3]{2a}$. **113.** $\sqrt[6]{32}\sqrt{2}$. **114.** $\sqrt[5]{3a}\sqrt{3a}$.

115. $\sqrt{\frac{a+1}{a-1}} + \frac{1}{1-a^2}\sqrt{4 - \frac{4}{a^2}}$. **116.** $\sqrt[3]{-x} - \sqrt[3]{-\frac{b^6}{8x^2}}$.

117. $b\sqrt{\frac{a}{b}} + a\sqrt{\frac{a}{b} + \frac{b}{a} + 2}$. **118.** $\frac{\sqrt[4]{4a^3}}{\sqrt[4]{a - 2ab^2 + ab^4}}$.

119. $\frac{2}{\sqrt[4]{9} - \sqrt[4]{4}}$. **120.** $\frac{3}{\sqrt[4]{9a^2} - \sqrt{b}}$. **121.** $\frac{\sqrt{3a} + 1}{\sqrt{a+b} - \sqrt{3a}}$.

122. $\sqrt[4]{x^{-2}y^{-2} - 6x^{-1}y^{-2} + 9y^{-2}}$. **123.** $\sqrt[4]{48x^4y} - \sqrt[4]{2a^4y} + \sqrt[8]{4y^2}$.

EXERCISE 64

Chapter Review

Find the value of each symbol, using Table I *if necessary.*

1. 6^{-3}. **2.** $(-2)^{-4}$. **3.** $(-15)^0$. **4.** $125^{\frac{1}{3}}$. **5.** $27^{\frac{1}{3}}$.

6. $4^{\frac{5}{2}}$. **7.** $25^{\frac{3}{2}}$. **8.** $9^{-\frac{3}{2}}$. **9.** $(-8)^{-\frac{2}{3}}$. **10.** $\sqrt{\frac{1}{7}}$.

11. $\sqrt{\frac{2}{5}}$. **12.** $(\sqrt{156})^2$. **13.** $(\sqrt[3]{239})^3$. **14.** $\sqrt{\frac{8}{45}}$. **15.** $\sqrt[3]{\frac{3}{16}}$.

16. $\sqrt[3]{\frac{4}{9}}$. **17.** $\sqrt{\frac{3}{32}}$. **18.** $\sqrt[3]{-\frac{5}{16}}$. **19.** $\sqrt{\sqrt[3]{4}}$. **20.** $\sqrt[3]{\sqrt{27}}$.

21. $\dfrac{2\sqrt{27}}{\sqrt{6}}$. **22.** $\dfrac{3\sqrt{15}}{\sqrt{75}}$. **23.** $\dfrac{\sqrt{2a}}{\sqrt{125a}}$. **24.** $\dfrac{\sqrt{50}}{\sqrt{25}}$. **25.** $\dfrac{1}{\sqrt[3]{-10}}$.

26. $\dfrac{3\sqrt{2}}{\sqrt{3}+\sqrt{2}}$. **27.** $\dfrac{\sqrt{3}-\sqrt{2}}{2\sqrt{2}-\sqrt{3}}$. **28.** $\dfrac{5^{\frac{1}{2}}-6^{\frac{1}{2}}}{5^{\frac{1}{2}}-3\sqrt{6}}$.

Write without fractions by use of negative exponents.

29. $\dfrac{3ab^2}{x^4a^3}$. **30.** $\dfrac{2hk^2}{5ay}$. **31.** $\dfrac{ab^4}{2a^{\frac{3}{2}}b^5}$. **32.** $\dfrac{3x}{a+2b}$.

Express without radicals, or negative signs or zero in the exponents, and simplify by use of the laws of exponents.

33. $\sqrt[3]{w^4}$. **34.** $\sqrt[5]{z^9}$. **35.** $\sqrt{9z}$. **36.** $\sqrt{4x^5}$. **37.** $\sqrt[3]{8z^2}$.

38. $\sqrt[3]{\frac{1}{8}x}$. **39.** $\sqrt[4]{9c^2}$. **40.** $\sqrt[10]{32a^5}$. **41.** $\sqrt[4]{(3b^3)^4}$. **42.** $(2ad^{\frac{3}{2}})^4$.

43. $8x^{-3}$. **44.** bd^{-5}. **45.** $(16x)^{\frac{3}{4}}$. **46.** $(ab^{-n})^k$. **47.** $(3a^0b^3)^2$.

48. $\left(\dfrac{5x}{3y^{\frac{1}{2}}}\right)^4$. **49.** $\left(\dfrac{8x^3}{27y^9}\right)^{\frac{2}{3}}$. **50.** $\dfrac{\sqrt[3]{4a^4x^3}}{\sqrt{2a^3x^5}}$. **51.** $\dfrac{\sqrt[3]{9ax^4}}{\sqrt[4]{9a^2x^2}}$.

52. $(2a+b^{-2})^{-1}$. **53.** $3(x^{-1}+2y)^{-1}$. **54.** $5(a^{-2}+3b^{-1})^{-2}$.

55. State the principal 4th root of 256 and principal cube root of -27.

Change to simplest radical form.

56. $\sqrt{96a^5}$. **57.** $\sqrt[5]{32xy^7}$. **58.** $\sqrt[3]{4x^{-3}}$. **59.** $\sqrt[3]{-32x^8}$.

60. $\sqrt{\frac{5}{14}}$. **61.** $\sqrt[3]{\frac{3}{32}}$. **62.** $\sqrt[5]{\frac{5}{16}}$. **63.** $\sqrt[3]{\frac{1}{4}} \div \sqrt{\frac{1}{3}}$.

64. $a^{\frac{2}{3}}b^{\frac{3}{2}}$. **65.** $4^{\frac{1}{3}}x^{\frac{5}{2}}$. **66.** $(\sqrt{2a})^3$. **67.** $(\sqrt[3]{5x})^4$.

68. $(2\sqrt[4]{3x})^3$. **69.** $\sqrt[3]{\sqrt{x}}$. **70.** $\sqrt{\sqrt[3]{4x^4}}$. **71.** $\sqrt[4]{81\sqrt[3]{x^2}}$.

72. $\sqrt[4]{\sqrt[3]{a^2}}$. **73.** $\sqrt[3]{\sqrt{b^9}}$. **74.** $\sqrt[4]{49}$. **75.** $\sqrt[6]{36}$.

76. $\sqrt{5}\sqrt[4]{5}$. **77.** $\sqrt[6]{216y^{-3}}$. **78.** $\sqrt{aw}\sqrt[6]{w^2}$. **79.** $\sqrt[3]{y^2}\sqrt{xy}$.

80. $\left(\dfrac{4w^5}{3z^3}\right)^{\frac{1}{2}}$ **81.** $\left(\dfrac{9z^4}{2x^3}\right)^{\frac{1}{2}}$. **82.** $\left(\dfrac{ab^4}{25z^7}\right)^{\frac{1}{3}}$. **83.** $\dfrac{\sqrt{\frac{1}{5}}}{\sqrt[3]{\frac{1}{3}}}$.

84. $3\sqrt{\tfrac{1}{2}} + 5\sqrt{\tfrac{1}{3}} + b\sqrt{8}$. **85.** $\sqrt{\tfrac{1}{3}} + 5\sqrt{12} - a\sqrt{108}$.

86. $(\sqrt{2} - 3\sqrt{a})^2$. **87.** $\sqrt[3]{-.125z^4x^{-2}}$. **88.** $(\sqrt{a} + 2\sqrt{b})^3$.

89. $\sqrt{2x - 3y} \div \sqrt[3]{4x^2 - 9y^2}$. **90.** $\sqrt{a^3 + 27b^3} \div \sqrt[3]{a + 3b}$.

91. $\dfrac{\sqrt{a+b} + \sqrt{a-b}}{\sqrt{a+b} - \sqrt{a-b}}$. **92.** $\dfrac{\sqrt{z} - \sqrt{z - 2w}}{3\sqrt{z} + \sqrt{z - 2w}}$.

93. $\sqrt[8]{81c^8x^4} - \sqrt[4]{9x^2z^4}$. **94.** $\sqrt[6]{64a^2x^6} - \sqrt[9]{a^3b^9}$.

95. $\sqrt{a + b} \div \sqrt[3]{a^2 - b^2}$.

96. $\sqrt{(a + b)^3} + 2a\sqrt[4]{a^2b^4 + 2ab^5 + b^6} + \sqrt{a^{-1} + b^{-1}}$.

97. $\sqrt{b^{-1} - a^{-2}} \div (\sqrt[3]{a - \sqrt{b}}\,\sqrt[3]{a + \sqrt{b}})$.

CHAPTER *10*

ELEMENTS OF QUADRATIC EQUATIONS

124. Terminology and foundation for imaginary numbers

By definition, R is a *square root* of -1 in case $R^2 = -1$. But, if R is either positive or negative, R^2 is *positive* and hence cannot equal -1. Obviously $R = 0$ is not a square root of -1. Hence, no real number is a square root of -1. Similarly, if P is positive, any square root R of the negative number $-P$ would satisfy the equation $R^2 = -P$. But, R^2 is positive or zero for all real values of R and hence $-P$ has no *real* number R as a square root. Therefore, in order that negative numbers may have square roots, we proceed to define numbers of a new type, to be called *imaginary* numbers.

Let the symbol $\sqrt{-1}$ be introduced as a new variety of number, called an imaginary number, with the property that *

$$\sqrt{-1}\sqrt{-1} = -1. \tag{1}$$

For convenience, we let $i = \sqrt{-1}$. Then, by definition $i \cdot i = -1$ or $i^2 = -1$. We agree † that the operations of addition, subtraction, and multiplication will be applied to combinations of i and real numbers as if i were an ordinary real literal number, with $i^2 = -1$. Then, in particular,

$$(-i)^2 = i^2 = -1, \tag{2}$$

so that $-i$, as well as $+i$, is a square root of -1. Any positive integral power of i can be easily computed by recalling that $i^2 = -1$ and hence

$$i^4 = (i^2)(i^2) = (-1)(-1) = 1. \tag{3}$$

* Refer to the introduction of negative numbers in Section 6 and observe the similarity of the present discussion.

† This procedure can be arrived at logically by a more advanced discussion.

ILLUSTRATION 1. $i^3 = i(i^2) = i(-1) = -i.$

$$i^{13} = i^{12}i = (i^4)^3 i = (1^3)(i) = i.$$

ILLUSTRATION 2. $(3 + 5i)(4 + i) = 12 + 23i + 5i^2$

$$= 12 + 23i - 5 = 7 + 23i.$$

If P is any positive number, we verify that

$$(i\sqrt{P})^2 = i^2P = -P; \quad (-i\sqrt{P})^2 = i^2P = -P.$$

Hence, the negative number $-P$ has the two square roots $\pm i\sqrt{P}$. Hereafter, we agree that the symbol $\sqrt{-P}$ or $(-P)^{\frac{1}{2}}$ represents the particular square root $i\sqrt{P}$. Then, $-P$ has the two square roots $\pm\sqrt{-P} = \pm i\sqrt{P}$. This agreement about the meaning of $\sqrt{-P}$ is equivalent to saying that *we should proceed as follows in dealing with the square root of a negative number:*

$$\sqrt{-P} = \sqrt{(-1)\cdot P} = \sqrt{-1}\sqrt{P} = i\sqrt{P}. \tag{4}$$

ILLUSTRATION 3. The square roots of -5 are

$$\pm\sqrt{-5} = \pm\sqrt{(-1)\cdot 5} = \pm\sqrt{-1}\sqrt{5} = \pm i\sqrt{5}.$$

$$\sqrt{\frac{-9a^4}{2}} = \sqrt{(-1)\cdot\frac{9a^4}{2}} = \sqrt{-1}\sqrt{\frac{9a^4\cdot 2}{2\cdot 2}} = \frac{3a^2i\sqrt{2}}{2}.$$

$$\sqrt{-\frac{64}{49}} = \sqrt{-1}\sqrt{\frac{64}{49}} = \frac{8}{7}i.$$

ILLUSTRATION 4. $\sqrt{-4}\sqrt{-9} = (i\sqrt{4})(i\sqrt{9}) = 6i^2 = -6.$

Note 1. The formula $\sqrt{a}\sqrt{b} = \sqrt{ab}$ was proved only for the case where $\sqrt{a}$ and $\sqrt{b}$ are real. We can verify that the formula does *not* hold if a and b are negative. Thus, by the formula,

$$\sqrt{-4}\sqrt{-9} = \sqrt{(-4)(-9)} = \sqrt{36} = 6,$$

which is wrong, because the correct result is -6 (in Illustration 4).

If a and b are *real* numbers, we call $(a + bi)$ a **complex number,** whose *real* part is a and *imaginary part* is bi. If $b \neq 0$, we call $(a + bi)$ an **imaginary number.** A **pure imaginary number** is one whose real part is *zero;* that is, $(a + bi)$ is a pure imaginary if $a = 0$ and $b \neq 0$. Any real number a is thought of as a complex number in which the coefficient of the imaginary part is zero; that is, $a = a + 0i$. In particular, 0 means $(0 + 0i)$.

ILLUSTRATION 5. $(2 - 3i)$ is an imaginary number. The real number 6 can be thought of as $(6 + 0i)$.

Note 2. In this book, unless otherwise stated, all literal numbers represent real numbers, except that hereafter i will always represent $\sqrt{-1}$. Any literal number in a radical of *even* order will be supposed *positive*, if this is possible and adds to our convenience.

★*Note 3.* The student has seen that we introduce imaginary numbers in order to provide square roots for negative numbers. It might then be inferred, incorrectly, that still other varieties of numbers would have to be introduced to provide cube roots, fourth roots, etc., of positive and negative numbers and also roots of all orders of imaginary numbers. An extremely interesting theorem is that the real numbers and the imaginary numbers, as just introduced, provide *all the numbers we need* in order to have at our disposal roots of all orders of any one of these numbers. Explicitly, in more advanced algebra,* it is proved that, if k is any positive integer, then any complex number N has just k distinct kth roots, which are also complex numbers (including real and imaginary numbers as special cases).

★*Note 4.* In the theory of electricity, it is customary to use j for $\sqrt{-1}$ because the letter i is reserved for a different purpose.

EXERCISE 65

Express by use of the imaginary unit i and simplify the remaining radical.

1. $\sqrt{-9}$. **2.** $\sqrt{-16}$. **3.** $\sqrt{-25}$. **4.** $\sqrt{-36}$. **5.** $\sqrt{-64}$.

6. $\sqrt{-28}$. **7.** $\sqrt{-50}$. **8.** $\sqrt{-23}$. **9.** $\sqrt{-7}$. **10.** $\sqrt{-32}$.

11. $\sqrt{-\frac{1}{4}}$. **12.** $\sqrt{-\frac{1}{9}}$. **13.** $\sqrt{-\frac{1}{49}}$. **14.** $\sqrt{-98}$. **15.** $\sqrt{-\frac{1}{25}}$.

16. $\sqrt{-.01}$. **17.** $\sqrt{-.09}$. **18.** $\sqrt{-\frac{4}{9}}$. **19.** $\sqrt{-\frac{16}{25}}$. **20.** $\sqrt{-.64}$.

21. $\sqrt{-.36}$. **22.** $\sqrt{-\frac{3}{2}}$. **23.** $\sqrt{-\frac{1}{5}}$. **24.** $\sqrt{-\frac{2}{7}}$. **25.** $\sqrt{-\frac{4}{11}}$.

26. $\sqrt{-a^2}$. **27.** $\sqrt{-4b^2}$. **28.** $\sqrt{-9c^2}$. **29.** $\sqrt{-a^2b^2}$.

30. $\sqrt{-16x^2}$. **31.** $\sqrt{-8x^2}$. **32.** $\sqrt{-12y^2}$. **33.** $\sqrt{-75h^2}$.

34. $\sqrt{-12w^3}$. **35.** $\sqrt{-128y^5x^4}$. **36.** $\sqrt{-4a^4y^3}$. **37.** $\sqrt{-27h^7}$.

38. $\sqrt{-\dfrac{a^2b^2}{4}}$. **39.** $\sqrt{-\dfrac{c^2d^2}{25}}$. **40.** $\sqrt{-\dfrac{4h^3}{36u^2}}$. **41.** $\sqrt{-\dfrac{9a^5}{4c^3}}$.

State the two square roots of each number.

42. -81. **43.** $-\frac{16}{25}$. **44.** $-\frac{49}{25}$. **45.** -63. **46.** $-\frac{4}{7}$.

* See De Moivre's Theorem and related topics in any college algebra.

Perform the indicated operation and simplify by use of $i^2 = -1$ *until* i *does not occur with an exponent greater than* 1.

47. i^5. **48.** i^7. **49.** i^6. **50.** i^8. **51.** i^{18}. **52.** i^{29}.

53. $(3 - i)(3 + i)$. **54.** $(3i + 5)(4 - 3i)$. **55.** $(3 + 2i)(3 - 2i)$.

56. $(2i + 3)^2$. **57.** $(3i - 2)(5i + 7)$. **58.** $(4i + 3)(-2i + 5)$.

59. $(5 - 2i)^2$. **60.** $(3i - 4)^2$. **61.** $(4i + 5)^2$.

62. $(2i^2 - 3i + 5)(2i - 3)$. **63.** $(i^3 - 2i^2 + 3)(3i^2 - 5)$.

64. $(4i - 7i^3)(2i + 5i^2)$. **65.** $(2i + 4i^4 - i^2)(2 + 3i)$.

66. $\sqrt{-2}\sqrt{-8}$. **67.** $\sqrt{-3}\sqrt{-75}$. **68.** $\sqrt{-27}\sqrt{-3}$.

69. $\sqrt{-2}(3 - 5\sqrt{-4})$. **70.** $\sqrt{-3}(5 - \sqrt{-27})$. **71.** $(5 - \sqrt{-8})^2$.

72. Substitute $x = 3 + 5i$ in $(x^2 - 6x + 34)$.

73. If $f(x) = 3x^2 + 2x - 7$, find $f(2i)$; $f(-3i)$; $f(2 - 5i)$.

74. If $f(x) = x^3 - x - 3$, find $f(2i)$; $f(1 + i)$.

Obtain an expression of the form $a + bi$ *for the fraction.*

75. $\dfrac{2 + 3i}{5 + 4i}$. **76.** $\dfrac{-5i + 1}{3i - 4}$. **77.** $\dfrac{3i + 4}{-i + 2}$. **78.** $\dfrac{2 + \sqrt{-4}}{3 - \sqrt{-9}}$.

HINT for Problem 75. Multiply numerator and denominator by $5 - 4i$.

125. Equations of the second degree

A **quadratic equation,** or an equation of the *second degree,* is an integral rational equation in which, after like terms are collected, the terms of highest degree in the variables are of the second degree. A quadratic equation in a variable x can be reduced to the standard form

$$ax^2 + bx + c = 0, \tag{1}$$

where a, b, and c are constants and $a \neq 0$. A *complete* quadratic equation in x is one for which $b \neq 0$, and a *pure* quadratic equation is one for which $b = 0$.

ILLUSTRATION 1. $3x^2 - 5x + 7 = 0$ is a complete quadratic equation and $5x^2 - 7 = 0$ is a pure quadratic in x.

126. Pure quadratic equations

To solve a pure quadratic equation in x, solve the equation for x^2 and extract square roots.

EXAMPLE 1. Solve: $7y^2 = 18 + 3y^2$.

SOLUTION. 1. $7y^2 - 3y^2 = 18; \quad 4y^2 = 18$.

2. Divide by 4: $y^2 = \frac{9}{2}$.

3. Hence, y must be a square root of $\frac{9}{2}$. On extracting square roots and using Table I, we obtain

$$y = \pm\sqrt{\frac{9}{2}} = \pm\sqrt{\frac{9\cdot 2}{2\cdot 2}} = \pm\frac{3}{2}\sqrt{2} = \pm\frac{3}{2}(1.414) = \pm 2.121.$$

EXAMPLE 2. Solve: $2y^2 + 35 = -5y^2$.

SOLUTION. $7y^2 = -35;\ y^2 = -5$. Hence, $y = \pm\sqrt{-5} = \pm i\sqrt{5}$.

EXAMPLE 3. Solve for x: $a^2x^2 + b^2 = abx^2 + a^2$.

SOLUTION. 1. $a^2x^2 - abx^2 = a^2 - b^2$.

2. Factor: $(a^2 - ab)x^2 = a^2 - b^2$.

3. Divide by $(a^2 - ab)$ and reduce to lowest terms:

$$x^2 = \frac{a^2 - b^2}{a^2 - ab} = \frac{(a-b)(a+b)}{a(a-b)};$$

or,

$$x^2 = \frac{a+b}{a}.$$

4. Extract square roots and rationalize the denominator:

$$x = \pm\sqrt{\frac{a+b}{a}} = \pm\frac{\sqrt{a(a+b)}}{a}.$$

Note 1. If the coefficients in a quadratic equation are explicit numbers and if a radical occurs in any solution which is a real number, always compute the decimal value of the solution by use of Table I. If it is desired to check such a solution, substitute the *radical* form instead of the decimal value, unless otherwise directed by the instructor. The approximate decimal value, as a rule, could not lead to an absolute check.

EXERCISE 66

Solve for x, or otherwise for the letter in the problem.

1. $5x^2 = 125$. **2.** $3x^2 = 12$. **3.** $x^2 = -9$. **4.** $4x^2 = -9$.

5. $9x^2 = -25$. **6.** $2x^2 = 3$. **7.** $5x^2 = 7$. **8.** $3x^2 = 11$.

9. $4x^2 = c$. **10.** $9x^2 = h$. **11.** $3ax^2 = h$. **12.** $2bx^2 = 16$.

13. $15 - 16z^2 = 4$. **14.** $9 - 7z^2 = 6$. **15.** $\frac{1}{2}x^2 - 1 = \frac{1}{3}x^2$.

16. $9x^2 + 49 = 0$. **17.** $7x^2 = 5 - 3x^2$. **18.** $\frac{2}{3}x^2 - \frac{3}{4} = \frac{1}{2}x^2$.

19. $18x^2 + 64 = 0.$ **20.** $\frac{3}{7}x^2 + 4 = \frac{2}{3}x^2.$ **21.** $\frac{1}{5}x^2 - \frac{3}{4} = \frac{1}{2}x^2.$

22. $4ax^2 - c = d.$ **23.** $4a + 2cx^2 = 4d.$ **24.** $9ax^2 - 4b = 9c^2.$

25. $4x^2 - 25a = 4bx^2.$ **26.** $9ax^2 = 4b + 9cx^2.$

27. $4x^2 + 25a = 25 + 4a^2x^2.$ **28.** $2cx^2 + 4d^2 = c^2 - 4dx^2.$

29. Solve $fr = mv^2$ for v. **30.** Solve $S = \frac{1}{2}gt^2$ for t.

31. Solve $A = \pi r^2$ for r. **32.** Solve $A = \frac{1}{2}\pi s^2 h$ for s.

Solve for x.

33. $\dfrac{x^2 + 2}{3} = \dfrac{5}{6}.$ **34.** $\dfrac{x}{3} = \dfrac{5}{4x}.$ **35.** $\dfrac{x}{4} - \dfrac{49}{x} = 0.$

36. $\dfrac{5}{x^2 - 6} = 3.$ **37.** $\dfrac{2x - 3}{4} = \dfrac{1}{2x + 3}.$ **38.** $\dfrac{ax - b}{2} = \dfrac{8}{ax + b}.$

127. Solution of an equation by factoring

We know that a product of two or more numbers equals *zero* when and only when *at least one of the factors is zero.** This fact is the basis for the following method, which applies to integral rational equations of any degree.

SUMMARY. *To solve an equation in x by use of factoring:*

1. *Clear the equation of fractions if necessary by multiplying both sides by the* LCD *of all fractions involved.*

2. *Transpose all terms to one member and thus obtain zero as the other member. Factor the first member if possible.*

3. *Place each factor equal to zero and solve for x.*

EXAMPLE 1. Solve: $$6 - 5x - 6x^2 = 0.$$

SOLUTION. 1. Multiply both sides by -1, to obtain convenience in factoring with a *positive* coefficient for x^2.

$$6x^2 + 5x - 6 = 0.$$

2. Factor: $$(3x - 2)(2x + 3) = 0.$$

3. The equation is satisfied if

$$3x - 2 = 0 \quad \text{or if} \quad 2x + 3 = 0.$$

4. If $3x - 2 = 0$, then $3x = 2$; $x = \frac{2}{3}$ is one solution.

5. If $2x + 3 = 0$, then $2x = -3$; $x = -\frac{3}{2}$ is a second solution.

* This fact holds for a product of complex numbers.

EXAMPLE 2. Solve: $4x^2 + 20x + 25 = 0.$

SOLUTION. 1. Factor:

$$(2x + 5)^2 = 0; \quad \text{or} \quad (2x + 5)(2x + 5) = 0.$$

2. If $2x + 5 = 0$, then $x = -\frac{5}{2}$. Since each factor gives the same value for x, we agree to say that the equation has *two equal roots.*

In solving an equation, if both sides are divided by an expression *involving the unknowns*, solutions may be *lost.*

EXAMPLE 3. Solve: $5x^2 = 8x.$

SOLUTION. 1. Transpose $8x$:

$$5x^2 - 8x = 0; \qquad x(5x - 8) = 0.$$

2. Hence, $x = 0$ or $5x - 8 = 0$; the solutions are 0 and $\frac{8}{5}$.

INCORRECT SOLUTION. Divide both sides of $5x^2 = 8x$ by x:

$$5x = 8.$$

Then, incorrectly, we obtain $x = \frac{8}{5}$ as the only solution. In this incorrect solution, the root $x = 0$ was lost on dividing by x.

Some literal quadratic equations can be solved by factoring.

EXAMPLE 4. Solve for x: $2a^2x^2 + 3abx - 2b^2 = 0.$

SOLUTION. 1. Factor: $(2ax - b)(ax + 2b) = 0.$

2. If $2ax - b = 0,$ then $2ax = b;$ $x = \dfrac{b}{2a}.$

3. If $ax + 2b = 0,$ then $ax = -2b;$ $x = -\dfrac{2b}{a}.$

4. The equation has the two solutions $\dfrac{b}{2a}$ and $-\dfrac{2b}{a}.$

EXERCISE 67

Solve by factoring.

1. $x^2 - 3x = 10.$	2. $y^2 - 5y = 14.$	3. $x^2 + x = 12.$
4. $x^2 + 3x = 28.$	5. $21x = 14x^2.$	6. $9x^2 - 144 = 0.$
7. $3x^2 - 7x = 0.$	8. $6x^2 = 15x.$	9. $5x^2 - 9x = 0.$
10. $x^2 + 8 = 6x.$	11. $4x^2 - 25 = 0.$	12. $x^2 + 15 = 8x.$
13. $2x^2 + 5x = 3.$	14. $3x^2 - 2x = 5.$	15. $8x^2 + 3 = 10x.$

16. $16x^2 = 24x - 9.$ **17.** $25y^2 = 20y - 4.$ **18.** $x^2 + 6x = -9.$

19. $4y^2 + 4y = -1.$ **20.** $3x^2 + 2 = -7x.$ **21.** $2x^2 + 7x = -6.$

22. $10x + 3 + 8x^2 = 0.$ **23.** $12 - 5x^2 - 17x = 0.$

24. $6x^2 - 19x + 15 = 0.$ **25.** $16x^2 + 40x + 25 = 0.$

26. $8 - 22x + 15x^2 = 0.$ **27.** $15 - 7w - 4w^2 = 0.$

28. $8x^2 + 2x - 15 = 0.$ **29.** $7x^2 + 9x - 10 = 0.$

30. $49x^2 + 28x = -4.$ **31.** $4 + 5x - 9x^2 = 0.$

32. $8 - 2x - x^2 = 0.$ **33.** $6 + 5x - 6x^2 = 0.$

Solve for x or w or z.

34. $3bx^2 + cx = 0.$ **35.** $2ax^2 - 3dx = 0.$

36. $x^2 + ax - 6a^2 = 0.$ **37.** $x^2 + 5bx + 6b^2 = 0.$

38. $2x^2 + bx - 3b^2 = 0.$ **39.** $3w^2 - bw - 4b^2 = 0.$

40. $4b^2x^2 + 4abx + a^2 = 0.$ **41.** $6b^2x^2 - 7bx - 3 = 0.$

42. $2a^2x^2 - abx - 3b^2 = 0.$ **43.** $9a^2x^2 + 12abx + 4b^2 = 0.$

44. $\dfrac{3}{4x^2} + \dfrac{7}{8x} - \dfrac{5}{2} = 0.$ **45.** $\dfrac{8}{3w + 3} + \dfrac{2 + 3w}{3w + 4} = -1.$

46. $\dfrac{5}{x + 4} - \dfrac{3}{x - 2} = 4.$ **47.** $\dfrac{2}{x - 1} - \dfrac{3}{2x + 5} = \dfrac{5}{3}.$

48. $\dfrac{z - 1}{z - 2} - \dfrac{11}{12} = \dfrac{1}{2z - 2}.$ **49.** $\dfrac{x}{x + 1} = \dfrac{2x}{1 - 4x} - \dfrac{1}{x + 1}.$

50. $\dfrac{10w}{2w - 1} = \dfrac{4w + 1}{2w - 2} - \dfrac{4w - 7}{2w - 1}.$ **51.** $\dfrac{2x + 11}{2x + 8} - \dfrac{3x - 1}{x - 1} = 0.$

52. $(x + 3)(2x - 5)(3x + 7) = 0.$ **53.** $6x^3 + x^2 - 15x = 0.$

54. $(2x - 3)(3x + 5) = 2x + 7.$ **55.** $(3x - 1)(2x + 5) = 3x + 19.$

128. Completing a square

A binomial $x^2 + px$ becomes a *perfect square* if we add *the square of one half of the coefficient of x.* That is, we complete a square if we add $\left(\dfrac{p}{2}\right)^2$ or $\dfrac{p^2}{4}$:

$$x^2 + px + \frac{p^2}{4} = \left(x + \frac{p}{2}\right)^2. \tag{1}$$

ILLUSTRATION 1. $x^2 - 6x$ becomes a perfect square if we add the square of $\frac{1}{2}(6)$, or 3:

$$x^2 - 6x + 9 = (x - 3)^2.$$

ILLUSTRATION 2. To make $x^2 - 7x$ a perfect square, we add $(\frac{7}{2})^2$ or $\frac{49}{4}$:

$$x^2 - 7x + \tfrac{49}{4} = (x - \tfrac{7}{2})^2.$$

SUMMARY. *To solve a quadratic equation in x by completing a square:*

1. *Transpose all terms involving x to the left side and all other terms to the right member and collect terms.*

2. *Divide both members by the coefficient of x^2.*

3. *Complete a square on the left by adding the square of one half of the absolute value of the coefficient of x to both sides.*

4. *Rewrite the left member as the square of a binomial.*

5. *Extract square roots, using the double sign on the right.*

EXAMPLE 1. Solve:
$$x^2 + 4x + 1 = 0. \tag{1}$$

SOLUTION. 1. Subtract 1:
$$x^2 + 4x = -1. \tag{2}$$
2. Since $4 \div 2 = 2$, add 2^2 or 4, to complete a square on the left:
$$x^2 + 4x + 4 = 4 - 1; \tag{3}$$
$$(x + 2)^2 = 3. \tag{4}$$
3. Extract square roots: $\quad x + 2 = \pm\sqrt{3}$, or
$$x = -2 \pm \sqrt{3}. \tag{5}$$

Thus, the roots are *irrational* numbers. To compute approximate values for the roots to three decimal places, we obtain $\sqrt{3}$ from Table I:

$$x = -2 + 1.732 = -.268 \qquad x = -2 - 1.732 = -3.732.$$

EXAMPLE 2. Solve:
$$3x^2 - 8x + 2 = 0.$$

SOLUTION. 1.
$$3x^2 - 8x = -2.$$
2. Divide by 3:
$$x^2 - \frac{8}{3}x = -\frac{2}{3}.$$

3. Since $\frac{8}{3} \div 2 = \frac{4}{3}$, add $(\frac{4}{3})^2$ or $\frac{16}{9}$ to complete a square.

$$x^2 - \frac{8}{3}x + \left(\frac{4}{3}\right)^2 = \frac{16}{9} - \frac{2}{3} = \frac{16}{9} - \frac{6}{9}.$$

Hence,
$$\left(x - \frac{4}{3}\right)^2 = \frac{10}{9}.$$

4. Extract square roots:

$$x - \frac{4}{3} = \pm\frac{\sqrt{10}}{3}; \qquad x = \frac{4 \pm \sqrt{10}}{3}. \tag{6}$$

From Table I, $\sqrt{10} = 3.162$. Hence,

$$x = \frac{4 + 3.162}{3} = 2.387, \quad \text{and} \quad x = \frac{4 - 3.162}{3} = .279.$$

EXAMPLE 3. Solve by completing a square: $x^2 + 4x + 7 = 0$.

SOLUTION. 1. $\qquad x^2 + 4x = -7.$

2. Since $(4 \div 2) = 2$, we add 2^2 or 4 to both sides:

$$x^2 + 4x + 4 = 4 - 7; \quad \text{or} \quad (x + 2)^2 = -3.$$

3. Hence, $\qquad x + 2 = \pm\sqrt{-3} = \pm i\sqrt{3};$

$$x = -2 \pm i\sqrt{3}.$$

EXAMPLE 4. Solve for x: $\qquad ax^2 + bx + c = 0.$

SOLUTION. 1. Subtract c: $\qquad ax^2 + bx = -c.$

2. Divide by a: $\qquad x^2 + \frac{b}{a}x = -\frac{c}{a}.$

3. Add $\left(\frac{b}{2a}\right)^2$, or $\frac{b^2}{4a^2}$: $\qquad x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}.$

Simplify: $\qquad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$

4. Extract square roots:

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}.$$

5. Subtract $\frac{b}{2a}$: $\qquad x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$

Note 1. An equality $A^2 = B^2$ is satisfied if $A = \pm B$, that is,

$$\textit{if} \quad A = B \quad \textit{or if} \quad A = -B. \tag{7}$$

Equally well, $A^2 = B^2$ if $-A = \pm B$, that is,

$$\textit{if} \quad -A = B \quad \textit{or if} \quad -A = -B. \tag{8}$$

If both sides of each equation in (8) are multiplied by -1, we obtain equations (7). Therefore, on extracting the square roots of both sides of $A^2 = B^2$, we obtain all possible information by writing $A = \pm B$, instead of writing $\pm A = \pm B$, where we read $\pm$ as "$+$ or $-$." That is, it is necessary to use the double sign $\pm$ on just *one* side if the square roots of both sides of an equation are extracted in solving it.

EXERCISE 68

Find what must be added to the expression to make it a perfect square, and then write this square.

1. $x^2 - 8x$.	**2.** $x^2 + 10x$.	**3.** $x^2 - 2cx$.	**4.** $x^2 + 4dx$.
5. $x^2 - \frac{4}{3}x$.	**6.** $x^2 + \frac{5}{2}x$.	**7.** $x^2 + \frac{7}{3}x$.	**8.** $x^2 - \frac{3}{4}x$.

Solve by completing a square.

9. $x^2 + 6x = 7$.	**10.** $x^2 + 10x + 24 = 0$.	**11.** $x^2 + 4x = 21$.
12. $x^2 + 9 = 6x$.	**13.** $x^2 + 4x + 4 = 0$.	**14.** $2w^2 + 3 = 8w$.
15. $2y^2 + 4y = 5$.	**16.** $x^2 + 13 = 6x$.	**17.** $x^2 + 5 = 4x$.
18. $5x^2 - 2 = 4x$.	**19.** $9x^2 + 1 = 12x$.	**20.** $9x^2 + 6x = 1$.
21. $4x^2 + 13 = 12x$.	**22.** $4x^2 + 4x - 3 = 0$.	**23.** $3x^2 + 8x = 1$.
24. $16x^2 + 9 = 24x$.	**25.** $9x^2 - 12x = 21$.	**26.** $4x^2 - 12x = 5$.

Verify the statement by substitution.

27. $x = (-2 + \sqrt{2})$ and $x = (-2 - \sqrt{2})$ satisfy $x^2 + 4x + 2 = 0$.

28. $x = (2 \pm 3i)$ are solutions of $x^2 - 4x + 13 = 0$.

Solve for x by completing a square.

29. $x^2 - 2ax = 15a^2$.	**30.** $x^2 + bx = 6b^2$.
31. $2x^2 - 5bx = 3b^2$.	**32.** $bx^2 - 4x - c = 0$.
33. $3x^2 + 2ax - b = 0$.	**34.** $.3x^2 - .06x - .144 = 0$.
35. $ax^2 + 4x - c = 0$.	**36.** $2x^2 + bx + c = 0$.
37. $Hx^2 + Kx + P = 0$.	**38.** $Ax^2 + 2Bx + C = 0$.

129. The quadratic formula

In Example 4 on page 179, the quadratic equation

$$ax^2 + bx + c = 0 \tag{1}$$

was solved by the method of completing a square; the solutions were found to be

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{2}$$

We call (2) the *quadratic formula*. In (2), it is permissible for a, b, and c to have any values, with $a \neq 0$.

SUMMARY. *To solve a quadratic equation in x by use of the quadratic formula:*

1. *Clear the equation of fractions and reduce it to the standard form* $ax^2 + bx + c = 0$.

2. *List the values of the coefficients a, b, and c.*

3. *Substitute the values of a, b, and c in the formula.*

ILLUSTRATION 1. To solve $3x^2 - 6x - 2 = 0$, we observe that $a = 3$, $b = -6$, and $c = -2$. Hence, from the quadratic formula,

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4\cdot 3\cdot(-2)}}{6} = \frac{6 \pm 2\sqrt{15}}{6} = \frac{3 \pm 3.873}{3};$$

$$x = \frac{3 + 3.873}{3} = 2.291, \quad \text{and} \quad x = \frac{3 - 3.873}{3} = -.291.$$

ILLUSTRATION 2. To solve $2x^2 - 4x + 5 = 0$, we notice that $a = 2$, $b = -4$, and $c = 5$. Hence, from the quadratic formula,

$$x = \frac{4 \pm \sqrt{16 - 40}}{4} = \frac{4 \pm \sqrt{-24}}{4} = \frac{4 \pm 2i\sqrt{6}}{4} = \frac{2 \pm i\sqrt{6}}{2}.$$

EXAMPLE 1. Solve for x: $\quad x^2 - 3ex + 5dx - 15de = 0.$

SOLUTION. 1. Group terms in x: $\quad x^2 + x(-3e + 5d) - 15de = 0.$

2. In the standard notation, $a = 1$, $b = -3e + 5d$, and $c = -15de$. From the formula,

$$x = \frac{-(-3e + 5d) \pm \sqrt{(5d - 3e)^2 - 4(-15de)}}{2}. \tag{3}$$

The radicand becomes

$$\begin{aligned} 25d^2 - 30de + 9e^2 + 60de &= 25d^2 + 30de + 9e^2 \\ &= (5d + 3e)^2. \end{aligned}$$

Hence, from (3),

$$x = \frac{-(-3e + 5d) \pm (5d + 3e)}{2};$$

$$x = \frac{3e - 5d + 5d + 3e}{2} = 3e; \qquad x = \frac{3e - 5d - 5d - 3e}{2} = -5d.$$

The solutions are $3e$ and $-5d$.

Note 1. In deriving the quadratic formula, we showed that an integral rational equation of the 2d degree in x has *just two roots* (which sometimes are identical). This result is a special case of a general theorem that, if the degree of the equation is n, the equation has *just n roots* (with repetitions of values possible among them).

EXERCISE 69

Solve by use of the quadratic formula. Check if directed by the instructor.*

1. $6x^2 + x - 2 = 0.$
2. $3y^2 + 2y - 5 = 0.$
3. $6y^2 - 7y - 3 = 0.$
4. $7y^2 - 8y = 12.$
5. $y^2 - 2y + 10 = 0.$
6. $x^2 + 13 = 4x.$
7. $4x^2 + 9 = 12x.$
8. $16x^2 - 25 = 0.$
9. $4x^2 - 8x + 1 = 0.$
10. $9x^2 + 6x + 1 = 0.$
11. $36x^2 - 49 = 0.$
12. $3y = 18 - 10y^2.$
13. $9x^2 + 6x = 1.$
14. $4 + 4x = 6x^2.$
15. $2x^2 + 3 = 8x.$
16. $2x^2 - 2x = 7.$
17. $4x^2 + 3 = 2x.$
18. $9x^2 + 6x = 7.$
19. $9x^2 + 16 = 0.$
20. $4x^2 + 13 = 4x.$
21. $3z^2 = 4z - 8.$
22. $4z^2 + 5 = 8z.$
23. $x^2 + .15 = .8x.$
24. $x^2 + 6 = 6x.$
25. $4x^2 + 53 = 4x.$
26. $3x^2 + 2x = 9.$
27. $4x^2 + 8x = -9.$
28. $9x^2 - 27 = 6x.$
29. $4x^2 + 13 = 12x.$
30. $18x^2 + 33x = 40.$
31. $25x^2 + 4 = 20x.$
32. $21x^2 + 19x = 12.$
33. $9y^2 + 23 = 30y.$
34. $24y^2 + 2y = 15.$
35. $4x^2 + 29 = -8x.$
36. $16x^2 + 34x = 15.$

Solve for x or y by use of the quadratic formula.

37. $6x^2 - 5dx - 6d^2 = 0.$
38. $2x^2 + hx - 15h^2 = 0.$
39. $ax^2 - dx + 3c = 0.$
40. $2ax^2 + 3bx - c = 0.$
41. $5ky^2 - 3ky + 6 = 0.$
42. $2hx^2 + 3x - 5h = 0.$
43. $y^2 + 2cy + dy + 2cd = 0.$
44. $y^2 - 4by + 3ay - 12ab = 0.$
45. $5kx^2 - 6 + 10kx - 3x = 0.$
46. $8hy^2 + 12y - 15 - 10hy = 0.$
47. $6hy^2 - 4hy + 10 - 15y = 0.$
48. $2x^2 - 3hx + h^2 - x = 1.$
49. $3x^2 + 3hx^2 - 6x + 5hx - 10 + 5x = 0.$

50. Check the solutions $(-b \pm \sqrt{b^2 - 4ac}) \div 2a$ by substitution in the equation $ax^2 + bx + c = 0.$

51. Solve for y in terms of x: $x^2 - 2y^2 - 2 + xy + x + 5y = 0.$

52. Solve for x in terms of y: $2y^2 + 15x^2 - 2 - x + 3y - 13xy = 0.$

53. Solve for x in terms of y in Problem 51.

54. Solve for y in terms of x in Problem 52.

* Table I is useful in detecting perfect square numbers. Thus, if we meet $\sqrt{1764}$, by reference to Table I we observe that $1764 = (42)^2$.

130. Outline for solution of quadratic equations

A pure quadratic equation should be solved by merely extracting square roots, as in Section 126. Any other quadratic equation should be solved by factoring if factors can be easily recognized. In all other cases, solve by use of the quadratic formula, unless otherwise specified. The method of completing the square is not recommended in any problem unless specifically requested; this method was introduced mainly as a means for deriving the quadratic formula.

131. Applications of quadratic equations

From geometry, we recall the Pythagorean theorem, which we associate with the triangle in Figure 12.

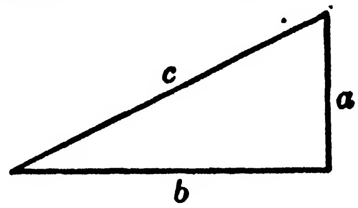

Fig. 12

If a and b are the lengths of the perpendicular sides and c is the length of the hypotenuse of a right angled triangle, then $a^2 + b^2 = c^2$.

Applications of the preceding theorem frequently introduce quadratic equations.

EXAMPLE 1. Find the length of a side of an equilateral triangle whose altitude is 3 feet shorter than a side.

SOLUTION. 1. In Figure 13, let ABC represent the triangle. Let x feet be the length of a side of $\triangle ABC$. Then, the lengths of AD and DC are, respectively, $\frac{1}{2}x$ and $(x - 3)$.

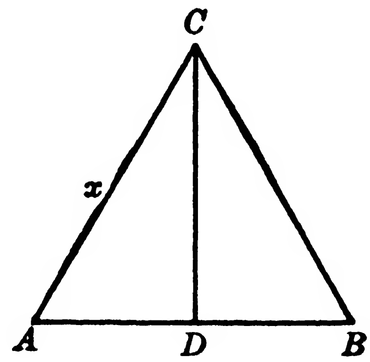

Fig. 13

2. From the Pythagorean theorem,

$$\overline{AC}^2 = \overline{AD}^2 + \overline{DC}^2,$$

or

$$x^2 = \left(\frac{x}{2}\right)^2 + (x - 3)^2. \qquad (1)$$

3. Simplify in (1):

$$x^2 = \frac{x^2}{4} + x^2 - 6x + 9;$$

$$x^2 - 24x + 36 = 0. \qquad (2)$$

4. Solve (2) by the quadratic formula:

$$x = \frac{24 \pm \sqrt{576 - 144}}{2} = 12 \pm 6\sqrt{3};$$

$$x = 22.39 \text{ and } 1.61. \qquad \text{(Using Table I)}$$

The smallest root has no significance in the problem because $(1.61 - 3)$ is negative. Hence, the side of the specified triangle is 22.39 feet long.

EXAMPLE 2. An airplane flies 560 miles against a head wind of 40 miles per hour. The plane took 28 minutes longer for this flight than would have been the case in still air. How fast could the airplane travel in still air?

INCOMPLETE SOLUTION. 1. Let x miles per hour be the speed of the airplane in still air. In flying against the wind, the speed is $(x - 40)$ miles per hour. From the equation *distance* = (*rate*)(*time*), the flight times for a distance of 560 miles against the wind and in still air are, respectively,

$$\frac{560}{x - 40} \quad \text{and} \quad \frac{560}{x}.$$

2. From the statement of the problem

$$\frac{560}{x - 40} = \frac{560}{x} + \frac{28}{60}.$$

The student should clear of fractions and solve for x.

MISCELLANEOUS EXERCISE 70

Solve each equation by three methods, (a) by factoring, (b) by completing a square, and (c) by use of the quadratic formula.

1. $2x^2 + 5x = 3.$ **2.** $3x^2 + 5 = 16x.$ **3.** $12x^2 + 11x = 15.$

Solve for x or y or z by the most convenient method.

4. $y^2 - 33 = 8y.$ **5.** $x^2 - 4x = 45.$ **6.** $16y^2 + 9 = 24y.$

7. $14x^2 - x = 3.$ **8.** $3z^2 - 6 = 2z.$ **9.** $49x^2 + 4 = 14x.$

10. $4y^2 + 17 = 12y.$ **11.** $1 + 25x^2 = 10x.$ **12.** $6x^2 + 5x = 56.$

13. $25x^2 - 20x = 1.$ **14.** $16 - 5x^2 = 0.$ **15.** $6x^2 = 7x.$

16. $6x^2 + 5 = 0.$ **17.** $9 - 11x^2 = 0.$ **18.** $4x^2 = 49x.$

19. $5y^2 + 36 = 0.$ **20.** $16y^2 + 1 = 8y.$ **21.** $20x^2 + 13x = 21.$

22. $\dfrac{8}{x} - \dfrac{5}{x + 3} = 3.$ **23.** $\dfrac{4}{3 - x} + \dfrac{2x}{5 + x} = 1.$

24. $\dfrac{3x}{x - 2} - \dfrac{1}{x^2 - 4} = 2.$ **25.** $\dfrac{7}{1 + 2y} - \dfrac{5y}{2y^2 + 3y + 1} = \dfrac{1}{3}.$

26. $\dfrac{1}{4} + \dfrac{2}{2x + 1} = \dfrac{1}{6x - 1}.$ **27.** $\dfrac{5 - x}{2 - x} - \dfrac{3 - 2x}{2x} = 1.$

28. $\frac{1}{2}gx^2 + ax = 3S.$ **29.** $4x^2 + 2bx + b = 1.$

30. $kx^2 - 2kx + 2 = x.$ **31.** $3x^2 + hx + 3kx + hk = 0.$

32. $ax^2 - 2bx = 2x + 3.$ **33.** $cx^2 + 2hx = 5 + 4kx.$

34. Solve for x by completing a square: $hx^2 + 2kx - m = 0$.

35. Solve for x by completing a square: $dx^2 - 3cx + h = 0$.

Solve each problem by introducing only one unknown number.

36. Divide 45 into two parts whose product is 434.

37. The area of a rectangle is 221 square feet and one side is 4 feet longer than the other. Find the dimensions.

38. Find two consecutive integers whose product is 306.

39. Find a number which is $\frac{273}{68}$ less than its reciprocal.

40. Find the length of a side of a square where a diagonal is 6 feet longer than a side.

41. Find the length of a side of an equilateral triangle whose altitude is 2 feet shorter than a side.

42. After plowing a uniform border inside a rectangular field 50 rods long by 40 rods wide, a farmer finds that he has plowed 60% of the field. Find the width of the border.

43. The diameter of a circular field is 40 yards. What increase in the diameter will increase the area by 440 square yards. (Use $\pi = 3\frac{1}{7}$.)

44. A circular field is surrounded by a cinder track whose width is 20 feet and area is $\frac{1}{3}$ of the area of the field. Find the radius of the field.

45. An airplane flew 660 miles in the direction of a wind and then took 40 minutes longer than on the outward trip to fly back against the same wind. If the plane flies at the rate of 200 miles per hour in still air, how fast was the wind blowing?

46. Jones travels 4 miles per hour faster than Smith and covers 224 miles in one hour less time than Smith. How fast does each man travel?

47. A motorboat takes 2 hours to travel 8 miles downstream and 4 miles back on a river which flows at the rate of 2 miles per hour. Find the rate at which the motorboat would travel in still water.

48. If A is the measured cross-section area of a chimney, its so-called effective area E is the smallest root of the equation $E^2 - 2AE + A^2 - .36A = 0$. Solve for E in terms of A and, from the result, find E if $A = 20$ square feet.

49. If an object is shot vertically from the surface of the earth with an initial velocity of v feet per second, and if air resistance and other disturbing factors are neglected, it is proved in physics that $s = vt - \frac{1}{2}gt^2$, where s feet is the height of the object above the surface at the end of t seconds and $g = 32$, approximately. (*a*) Solve for t in terms of s. (*b*) If $v = 200$ feet, use Part *a* to find when $s = 500$ feet and $s = 0$ feet.

CHAPTER 11

ADVANCED TOPICS IN QUADRATIC EQUATIONS

132. Graph of a quadratic function

A *quadratic function* of x is a polynomial of the *second* degree in x and hence has the form

$$ax^2 + bx + c,$$

where a, b, and c are constants and $a \neq 0$.

EXAMPLE 1. Graph the function $x^2 - 2x - 3$.

SOLUTION. Let $y = x^2 - 2x - 3$. We select values for x and compute the corresponding values for y. In Figure 14, we plot the points $(-3, 12)$, $(-2, 5)$, etc. In the table of values, we arrange the values of x in their *natural order* as they appear on the x-axis, because then the corresponding points on the graph are met in their natural order as we draw the curve. The curve through the plotted points is the graph of the function and is called a **parabola.** The point V at the rounded end is called the **vertex** of the parabola. Since V is the *lowest* point of the graph, the ordinate of V, or -4, is the *smallest* or **minimum value of the function,** and we call V the **minimum point** of the graph. The vertical line through V is called the **axis** of the parabola. The part of the curve to the right of this axis has exactly the same shape as the part to the left. That is, the parabola is *symmetrical* with respect to its axis. The equation of the axis of the parabola $y = x^2 - 2x - 3$ shown in Figure 14 is $x = 1$.

$x =$	-3	-2	0	1	2	4	5
$y =$	12	5	-3	-4	-3	5	12

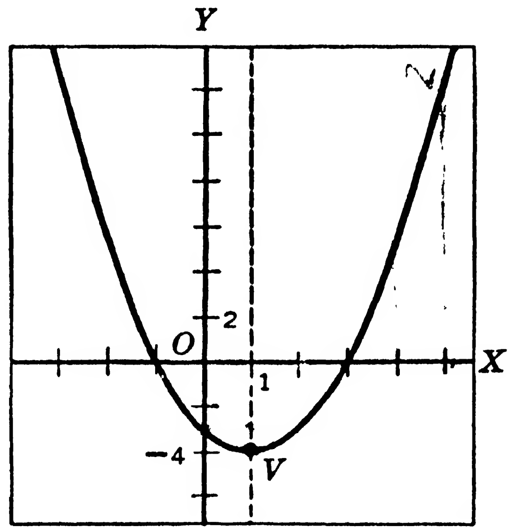

Fig. 14

If a parabola is concave downward (open downward), instead of concave upward as shown in Figure 14, then the vertex of the parabola is its *highest* point and is called the **maximum point** of the curve.

At a more advanced stage, we meet proofs of the following facts:

I. *The graph of* $ax^2 + bx + c$ *is a parabola with its axis perpendicular to the x-axis; this parabola is* **concave upward if** a **is positive** *and* **concave downward if** a **is negative.**

II. *The abscissa of the vertex of the parabola is* $x = -\dfrac{b}{2a}$; *when* x *has this value, the function has its* **minimum** *or its* **maximum** *value according as* a *is* **positive** *or* **negative.**

ILLUSTRATION 1. In Figure 14, at V, $x = -\dfrac{-2}{2 \cdot 1} = 1$.

SUMMARY. *To form a table of values in graphing a quadratic function* $f(x)$:

1. *Find the coordinates of the vertex of the graph.*

2. *Choose pairs of values of* x *where, in each pair, the values are equidistant from the vertex, one value on each side; the values of* $f(x)$ *corresponding to each pair will be equal.*

ILLUSTRATION 2. In Example 1, the abscissa of V is $x = 1$. Then, we selected $x = 1 \pm 1$, or $x = 2$ and $x = 0$; $x = 1 \pm 3$, or $x = 4$ and $x = -2$; etc. The corresponding pairs of values of y are equal.

★EXAMPLE 2. Divide 50 into two parts such that their product will be a maximum.

SOLUTION. 1. Let x be one part; $(50 - x)$ is the other part.

2. Let $f(x)$ represent the product $x(50 - x)$, or $f(x) = 50x - x^2$.

3. The maximum of $f(x)$ is attained at the *vertex* of the parabola which is the graph of $f(x)$, or when $x = -[50 \div (-2)] = 25$. Hence, the product of the parts of 50 will be *greatest* when they are *equal*, each being 25. The corresponding largest product is 625.

Note 1. After having formed a table of values for graphing a quadratic function of x, select the scales on the coordinate axes with care. Choose the unit for distance on the x-axis large enough to spread out the parabola in order to make it generously open. Choose the vertical unit *independently of the previous choice* of the x-unit in order to be able to plot all points from the table of values on the available part of the cross-section sheet.

EXERCISE 71

For each function, (a) find the coordinates of the vertex of the graph and the equation of its axis; (b) graph the function, with values of x extending at least 4 units on each side of the vertex; (c) state the maximum or minimum value of each function.

1. x^2. **2.** $4x^2$. **3.** $-x^2$. **4.** $-6x^2$. **5.** x^2+5.

6. x^2-4. **7.** x^2+6x+5. **8.** x^2-4x+7.

9. $-2x^2+4x+3$. **10.** $-3x^2+12x$. **11.** $2x^2+8x+3$.

12. $-3x^2+6x-5$. **13.** $4x^2-12x$. **14.** $2x^2-20x+4$.

State whether the function has a maximum or a minimum value, and obtain this value without graphing by finding the coordinates of the vertex.

15. $4x^2-16x+3$. **16.** $-3x^2+24x-7$. **17.** $-6x^2+8$.

18. If an object is shot vertically upward from the earth's surface with an initial velocity of 96 feet per second, (a) draw a graph of the distance s as a function of t; (b) from the graph, find when the object commences to fall, the maximum height which it reaches, and when it hits the surface. (Recall the formula of Problem 49, page 185.)

19. If an object is shot vertically upward from the earth's surface with an initial velocity of 80 feet per second, find when the object reaches its maximum elevation, without graphing. (See Problem 18.)

20. Graph the function $x^3-12x+3$ by use of the integral values of x from -4 to 4 inclusive.

★*Graph each of the following functions, with enough computed points to obtain a graceful curve.*

21. x^3. **22.** x^4. **23.** $-x^4$. **24.** $-x^3$.

25. x^3+2x^2-4x+3. **26.** $-3x^4-4x^3+12x^2+6$.

★*Solve each problem by introducing just one unknown x and then finding the maximum of a quadratic function of x, without graphing.*

27. Divide 60 into two parts whose product is a maximum.

28. Find the dimensions of the rectangular field of largest area which can be inclosed with 600 feet of wire fence.

29. In forming a trough with a rectangular cross section and open top, a long sheet of tin is bent upward on each long side. If the sheet is 30 inches wide, find the dimensions of the cross section with the largest possible area.

30. Divide H into two parts whose product is a maximum.

133. Graphical solution of an equation

If x has a value for which the graph of $f(x)$ meets the x-axis, then with this value of x we have $f(x) = 0$. Hence we are led to the following procedure.

SUMMARY. *To find approximate values of the real roots of an equation in x graphically:*

1. *Simplify and transpose all terms to one member to obtain an equation of the form $f(x) = 0$.*

2. *Graph the function $f(x)$ and measure the abscissas of the points where the graph meets the x-axis; each of these abscissas satisfies the equation $f(x) = 0$.*

EXAMPLE 1. Solve $x^2 - 2x - 3 = 0$ graphically.

SOLUTION. 1. Let $y = x^2 - 2x - 3$ and consider its graph in Figure 14, page 186. The graph crosses the x-axis at $x = 3$ and $x = -1$.

2. Since $y = 0$ when $x = 3$ and when $x = -1$, these are values of x for which $x^2 - 2x - 3 = 0$. That is, 3 and -1 are roots of the equation.

If the roots of $f(x) = 0$ are *imaginary*, this would be indicated by the fact that the graph of $f(x)$ would *not* meet the x-axis.

134. Graphical solution of a quadratic equation

In order to solve the equation

$$ax^2 + bx + c = 0 \tag{1}$$

graphically, we construct the graph of the quadratic function

$$ax^2 + bx + c. \tag{2}$$

The parabola, which is the graph of this function,

I. *cuts the x-axis in two points when and only when equation* 1 *has unequal real roots;*

II. *touches the x-axis in just one point, or is tangent to the x-axis, when and only when the roots are equal;*

III. *does not meet the x-axis when and only when the roots are imaginary.*

Note 1. A parabola can be defined geometrically as the curve of intersection when a right circular cone is cut by a plane which is parallel to a straight line on the cone through its apex.

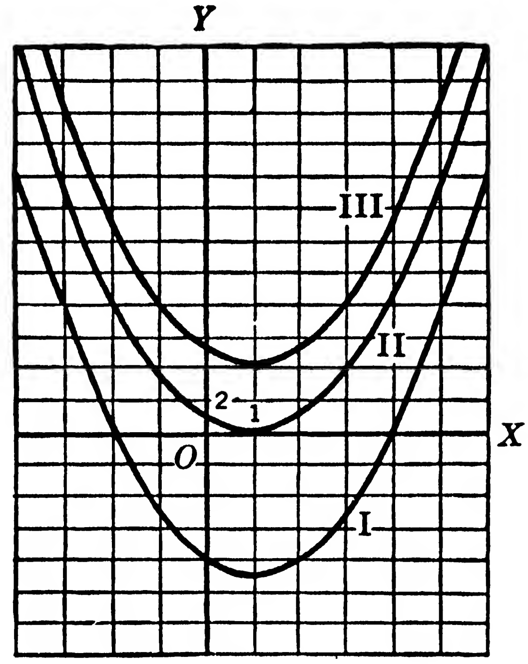

Fig. 15

ILLUSTRATION 1. In Figure 15, parabolas I, II, and III are, respectively, the graphs of the functions in the left members of the following equations.

$$\text{(I)}\quad x^2 - 2x - 8 = 0;$$

$$\text{(II)}\quad x^2 - 2x + 1 = 0;$$

$$\text{(III)}\quad x^2 - 2x + 5 = 0.$$

From the graphs, we see that (I) has the roots $x = 4$ and $x = -2$; (II) has equal roots, $x = 1$; (III) has imaginary roots.

In graphing a specified quadratic *function*, we have no license to simplify its form by multiplication or by division. But, before solving a quadratic *equation* graphically, we may (1) clear the equation of fractions; (2) divide out any common constant factor from all terms; (3) make the coefficient of x^2 positive. Operation 3 would cause the corresponding graph to open upward.

EXERCISE 72

Find the real roots of the equation graphically.

1. $2x - 5 = 0.$ **2.** $x^2 + 2x - 8 = 0.$ **3.** $x^2 + 6x + 9 = 0.$

4. $x^2 + 4x + 7 = 0.$ **5.** $x^2 - 2x + 3 = 0.$ **6.** $\frac{5}{2}z = 2 - z^2.$

7. $5w = 13 - 2w^2.$ **8.** $4x = 2x^2 - 5.$ **9.** $2x = -\frac{10}{3} - \frac{1}{3}x^2.$

10. $4x^2 = 12x - 9.$ **11.** $2x^2 = 4x + 3.$ **12.** $3x^2 = 6x - 5.$

Draw the graph of $x^2 + 4x + 4$ *and then, in Problems* 13 *and* 14, *find the specified results by use of the graph.*

13. Find the values of x for which the value of the function is 1.

14. Solve $x^2 + 4x + 4 = 6$ by inspection of the graph.

15. By use of a single graph, solve each of the following equations graphically: $2x^2 - 5x = 0$; $2x^2 - 5x + 3 = 0$; $2x^2 - 5x - 7 = 0$.

135. Character of the roots

Let r and s represent the roots of $ax^2 + bx + c = 0$. Then, from the quadratic formula,

$$r = \frac{-b + \sqrt{b^2 - 4ac}}{2a}; \qquad s = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \tag{1}$$

We assume that a, b, and c are real numbers and that $a \neq 0$. Then, the roots are *imaginary* when and only when $b^2 - 4ac$ is *negative;* if one root is imaginary, the other is also.

If $b^2 - 4ac = 0$, then $r = s = -b/2a$. Moreover, if $r = s$, on subtracting the expressions in (1) we obtain

$$0 = r - s = \frac{2\sqrt{b^2 - 4ac}}{2a}; \qquad \frac{\sqrt{b^2 - 4ac}}{a} = 0; \qquad \sqrt{b^2 - 4ac} = 0.$$

Hence, if $r = s$ then $b^2 - 4ac = 0$.

From the preceding remarks and Section 134, we see that the items in any row of the following summary hold simultaneously.

THE ROOTS OF $ax^2 + bx + c = 0$	THE VALUE OF $b^2 - 4ac$	THE GRAPH OF $ax^2 + bx + c$
real and unequal	$b^2 - 4ac > 0$	*cuts x-axis in two points*
real and equal	$b^2 - 4ac = 0$	*is tangent to x-axis*
imaginary	$b^2 - 4ac < 0$	*does not touch x-axis*

If a, b, and c are *rational* numbers, the roots are rational when and only when $\sqrt{b^2 - 4ac}$ is *real* and is a rational number. That is, *the roots are* **rational** *numbers when and only when $b^2 - 4ac$ is a* **perfect square.**

We call $b^2 - 4ac$ the **discriminant** of the *quadratic equation* $ax^2 + bx + c = 0$, or of the *quadratic function* $ax^2 + bx + c$, because, as soon as we know the value of $b^2 - 4ac$, we can tell the general character of the roots of the equation without solving it, and the general nature of the graph of the function without graphing it.

ILLUSTRATIONS OF THE USE OF THE DISCRIMINANT

EQUATION	DISCRIMINANT	HENCE, THE ROOTS ARE
$4x^2 - 3x + 5 = 0$	$(-3)^2 - 4\cdot4\cdot5 = -71$	*imaginary numbers*
$4x^2 - 4x + 1 = 0$	$(-4)^2 - 4\cdot4 = 0$	*real; equal; rational*
$4x^2 - 3x - 5 = 0$	$(-3)^2 + 4\cdot4\cdot5 = 89$	*real; unequal; irrational*
$x^2 - 2x - 3 = 0$	$(-2)^2 - 4(-3) = 16 = 4^2$	*real; unequal; rational*

Before computing the discriminant in any equation, it should be simplified by clearing of fractions and combining terms.

EXAMPLE 1. State what you can learn about the graph of the quadratic function $-3x^2 + 5x - 6$ *without* graphing.

SOLUTION. 1. The discriminant of the function is $25 - 72 = -47$.

2. Hence, the graph would *not* touch the x-axis. Since the coefficient of x^2 is -3, the graph is concave *downward* and therefore must lie wholly *below* the x-axis.

136. Conjugate imaginaries

If two imaginary numbers differ only in the *signs of the coefficients of their imaginary parts,* then either of the given numbers is called the *conjugate* of the other.

ILLUSTRATION 1. The conjugate of $(3 + 5i)$ is $(3 - 5i)$. The conjugate of $(a + bi)$ is $(a - bi)$.

When the roots of a quadratic equation are imaginary, *these roots are conjugate imaginary numbers,* because the imaginary parts come from $\pm\sqrt{b^2 - 4ac}$ in the quadratic formula.

ILLUSTRATION 2. The roots of $x^2 + 4x + 5 = 0$ are

$$x = \frac{-4 \pm \sqrt{16 - 20}}{2} = -2 \pm i, \text{ conjugate imaginaries.}$$

EXERCISE 73

Compute the discriminant and tell the character of the roots, without solving.

1. $y^2 - 7y + 10 = 0$. **2.** $y^2 - 4y - 21 = 0$. **3.** $x^2 + 2x - 2 = 0$.

4. $3x^2 - 5x + 7 = 0$. **5.** $9x^2 + 12x + 4 = 0$. **6.** $4x^2 + 4x = 3$.

7. $30 - 9y^2 = 25y$. **8.** $3x - 2 = 5x^2$. **9.** $25 + 4x^2 = 20x$.

10. $2x - 3 = 6x^2$. **11.** $5x^2 + 1 = 2x$. **12.** $25x^2 + 1 = -10x$.

13. $8x^2 - 7 = 0$. **14.** $5x^2 - 3x = 0$. **15.** $1 = 2x - 2x^2$.

16. $3 + 5x^2 = 0$. **17.** $6x = 9x^2 + 4$. **18.** $x^2 + .4x + .3 = 0$.

Solve graphically; check the graph by computing the discriminant and thus determining the character of the roots.

19. $x^2 - 4x = 6$. **20.** $x^2 + 7 = 4x$. **21.** $4x^2 + 4x = 1$.

Compute the discriminant of the function and, without graphing, state all facts which you can learn about its graph.

22. $4x^2 - 12x + 9$. **23.** $2x^2 - 3x - 5$. **24.** $3x^2 - 4x$.

25. $-3x^2 + 5x - 7$. **26.** $4x^2 + 5x + 7$. **27.** $-3x^2 - 2x + 4$.

Specify the conjugate number for the imaginary number.

28. $3 + 7i$. **29.** $-2 - 5i$. **30.** $-2 + \sqrt{-9}$. **31.** $6\sqrt{-1}$.

137. Sum and product of the roots

By use of

$$r = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad s = \frac{-b - \sqrt{b^2 - 4ac}}{2a},$$

we obtain
$$r + s = \frac{-2b}{2a} = -\frac{b}{a};$$

$$rs = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \cdot \frac{-b - \sqrt{b^2 - 4ac}}{2a};$$

$$rs = \frac{(-b)^2 - (b^2 - 4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}.$$

Hence, for the equation $ax^2 + bx + c = 0$,

the sum of the roots equals $-\dfrac{b}{a}$: $\qquad \boldsymbol{r + s = -\dfrac{b}{a}}$. (1)

the product of the roots equals $\dfrac{c}{a}$: $\qquad \boldsymbol{rs = \dfrac{c}{a}}$. (2)

ILLUSTRATION 1. For $3x^2 - 5x + 7 = 0$, we find $r + s = \frac{5}{3}$ and $rs = \frac{7}{3}$.

138. Factored form of a quadratic function

THEOREM I. *If r and s are the roots of $ax^2 + bx + c = 0$,*

$$\boldsymbol{ax^2 + bx + c = a(x - r)(x - s).} \tag{1}$$

Proof. 1. We can write

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right).$$

2. From Section 137, $\dfrac{b}{a} = -(r + s)$ and $\dfrac{c}{a} = rs$. Hence,

$$ax^2 + bx + c = a[x^2 - (r + s)x + rs] = a(x - r)(x - s).$$

ILLUSTRATION 1. A quadratic equation whose roots are 5 and -3 is

$$(x+3)(x-5)=0, \quad \text{or} \quad x^2-2x-15=0. \qquad [a=1 \text{ in } (1)]$$

ILLUSTRATION 2. A quadratic equation whose roots are $\frac{1}{2}(2 \pm 3i)$ is

$$a[x-\tfrac{1}{2}(2+3i)][x-\tfrac{1}{2}(2-3i)]=0.$$

To eliminate fractions we use $a = 4 = 2 \cdot 2$, and then group within parentheses to exhibit the sum and difference of two quantities, as an aid in multiplying:

$$2\left(x-\frac{2+3i}{2}\right)\cdot 2\left(x-\frac{2-3i}{2}\right)=(2x-2-3i)(2x-2+3i)=0;$$

$$[(2x-2)-3i][(2x-2)+3i]=0, \quad \text{or} \quad (2x-2)^2-9i^2=0.$$

Since $i^2 = -1$, we have

$$4x^2-8x+4+9=0, \quad \text{or} \quad 4x^2-8x+13=0.$$

Under certain circumstances, we have seen how to solve the equation $ax^2+bx+c=0$ by first factoring the function ax^2+bx+c. Formula 1 permits us to *reverse* this process and *to factor the function by first solving the equation* (of course, *not* using factoring in the solution).

EXAMPLE 1. Factor $6x^2 - 23x + 20$ by first solving an equation.

SOLUTION. 1. Solve $6x^2 - 23x + 20 = 0$, by the quadratic formula:

$$x=\frac{23 \pm \sqrt{49}}{12}=\frac{23 \pm 7}{12};$$

$$x=\frac{5}{2} \quad \text{and} \quad x=\frac{4}{3}.$$

2. From formula 1:

$$6x^2-23x+20=6(x-\tfrac{5}{2})(x-\tfrac{4}{3})=(2x-5)(3x-4).$$

Formula 1 states that *any quadratic function of x can be expressed as a product of factors which are linear in x.* However, these factors involve rational, irrational, or imaginary coefficients depending on the nature of the roots r and s. In particular, from the facts about *rational* roots, on page 191, we draw the following conclusion:

If a, b, and c are rational numbers, ax^2+bx+c can be expressed as a product of real linear factors with rational coefficients when and only when the discriminant b^2-4ac is a perfect square.

EXERCISE 74

Find the sum and the product of the roots of each equation, in the unknown x, without solving for x.

1. $x^2 + 5x - 3 = 0$.
2. $2x = 5x^2 + 7$.
3. $4x^2 = 3x - 6$.
4. $7 - 5x = 2x^2$.
5. $7 - 3x = 4x^2$.
6. $5x^2 - 17 = 0$.
7. $18 = 5x^2$.
8. $12x^2 + 3 = 0$.
9. $5 - 9x^2 = 7x$.
10. $2x^2 - 3 = 5x$.
11. $5 = 7x^2 - 4x$.
12. $4x^2 - 12x = 9$.
13. $ax^2 + dx = h$.
14. $cx^2 = 3x - b$.
15. $4x^2 = ax + c$.
16. $2x^2 + 3x + ax + c = 0$.
17. $5x^2 + ax^2 + 3x + d = 0$.
18. $2x^2 + 3x + 2a + d = 0$.
19. $x^2 + cx^2 - 2x + cx = d - 5$.

Compute the indicated product.

20. $4(x - \frac{3}{2})(x + \frac{5}{2})$.
21. $6(x - \frac{1}{3})(x + \frac{3}{2})$.
22. $(x - 2 + 3i)(x - 2 - 3i)$.
23. $(x + 1 - 2\sqrt{2})(x + 1 + 2\sqrt{2})$.

Form a quadratic equation with integral coefficients having the given numbers as roots.

24. $3;\ -7$.
25. $-2;\ -3$.
26. $\frac{1}{4};\ -\frac{3}{2}$.
27. $\frac{1}{3};\ 2$.
28. $-\frac{5}{7};\ -\frac{3}{4}$.
29. $2;\ -\frac{5}{3}$.
30. $-\frac{3}{4};\ -\frac{5}{2}$.
31. $\pm\sqrt{2}$.
32. $\pm\frac{1}{2}\sqrt{3}$.
33. $\pm 3\sqrt{2}$.
34. $\pm 2i$.
35. $\pm\frac{2}{3}i$.
36. $1 \pm \sqrt{2}$.
37. $-2 \pm \sqrt{5}$.
38. $3 \pm 2\sqrt{2}$.
39. $\frac{1}{2} \pm \frac{3}{2}\sqrt{3}$.
40. $-\frac{1}{3} \pm \frac{1}{3}\sqrt{2}$.
41. $3 \pm 5i$.
42. $-2 \pm 3i$.
43. $4 \pm 2i$.
44. $-\frac{2}{3} \pm \frac{1}{3}i$.
45. $2 \pm 2i\sqrt{5}$.
46. $\frac{1}{2} \pm \frac{3}{2}i\sqrt{3}$.
47. $-\frac{2}{3} \pm \frac{1}{3}i\sqrt{2}$.

Factor, after first solving a related quadratic equation by use of the quadratic formula.

48. $12x^2 + 11x - 36$.
49. $27x^2 + 21x - 40$.
50. $27x^2 - 6xy - 16y^2$.
51. $24x^2 - 13x - 60$.
52. $48x^2 + 50x - 75$.
53. $27x^2 + 12x - 32$.

Without factoring or solving any equation, determine whether or not the expression has real linear factors with rational coefficients.

54. $8x^2 + 7x - 2$.
55. $11x^2 + 12x - 5$.
56. $6x^2 + 25xy + 25y^2$.

★*Factor, perhaps by use of imaginary or irrational numbers.*

57. $x^2 + 6x + 10$.
58. $4x^2 - 12x + 7$.
59. $2x^2 - 2x + 5$.

139. Equations in quadratic form

EXAMPLE 1. Solve: $$x^4 - 5x^2 + 6 = 0. \tag{1}$$

SOLUTION. 1. Factor: $$(x^2 - 3)(x^2 - 2) = 0.$$

2. If $x^2 - 3 = 0$, then $x = \pm\sqrt{3}$; if $x^2 - 2 = 0$, then $x = \pm\sqrt{2}$. The given equation has four solutions, $\pm\sqrt{3}$ and $\pm\sqrt{2}$.

SECOND SOLUTION. 1. Let $y = x^2$; then $y^2 = x^4$ and, from (1),
$$y^2 - 5y + 6 = 0.$$

2. Solve for y:
$$(y - 3)(y - 2) = 0;$$
hence, $y = 3$ and $y = 2$.

3. If $y = 2$, then $x^2 = 2$ and $x = \pm\sqrt{2}$.

4. If $y = 3$, then $x^2 = 3$ and $x = \pm\sqrt{3}$.

Comment. The given equation is said to be in the *quadratic form* in x^2 because we obtain a quadratic in y on substituting $y = x^2$.

EXAMPLE 2. Solve: $2x^{-4} - x^{-2} - 3 = 0.$

SOLUTION. 1. Let $y = x^{-2}$; then $y^2 = x^{-4}$ and $2y^2 - y - 3 = 0$.

2. Solve for y: $$(2y - 3)(y + 1) = 0;$$
hence, $y = -1$ and $y = \frac{3}{2}$.

3. If $y = \dfrac{3}{2}$, then $x^{-2} = \dfrac{3}{2}$; $\dfrac{1}{x^2} = \dfrac{3}{2}$; $x^2 = \dfrac{2}{3}$; $x = \pm\dfrac{1}{3}\sqrt{6}$.

4. If $y = -1$, then $x^{-2} = -1$; $\dfrac{1}{x^2} = -1$; $x^2 = -1$; $x = \pm i$.

5. The solutions are $\pm i$ and $\pm\frac{1}{3}\sqrt{6}$.

EXAMPLE 3. Solve: $(x^2 + 3x)^2 - 3x^2 - 9x - 4 = 0.$

INCOMPLETE SOLUTION.

1. Group terms: $$(x^2 + 3x)^2 - 3(x^2 + 3x) - 4 = 0.$$

2. Let $y = x^2 + 3x$; then $y^2 - 3y - 4 = 0$; hence, $y = 4$, and $y = -1$. We should then solve
$$x^2 + 3x = 4 \quad \text{and} \quad x^2 + 3x = -1.$$

Note 1. In solving an equation of the form $x^k = A$ where k is a positive integer greater than 2, we agree for the present that we desire only *real* solutions *unless otherwise specified.* The real solutions, if any, of $x^k = A$ are the real kth roots of A. Thus, $x^4 = -8$ has no real solutions while $x^6 = 64$ has the real solutions $x = \pm\sqrt[6]{64} = \pm 2$.

EXAMPLE 4. Obtain *all* roots by use of factoring: $8x^3 + 125 = 0$.

SOLUTION. 1. Factor: $(2x + 5)(4x^2 - 10x + 25) = 0$.

2. Hence, $2x + 5 = 0$, or $4x^2 - 10x + 25 = 0$.

3. The solutions are

$$x = -\tfrac{5}{2} \quad \text{and} \quad x = \tfrac{1}{8}(10 \pm \sqrt{100 - 400}) = \tfrac{5}{4} \pm \tfrac{5}{4}i\sqrt{3}.$$

EXAMPLE 5. Find the four 4th roots of 625.

SOLUTION. 1. If x is any 4th root of 625, then $x^4 = 625$.

2. Solve for x: $x^4 - 625 = 0$;

$$(x^2 - 25)(x^2 + 25) = 0; \quad x^2 = 25 \quad \text{or} \quad x^2 = -25.$$

Hence, $x = \pm 5$ and $x = \pm 5i$ are the desired 4th roots of 625.

Note 2. In this section the student has met further illustrations of the truth of the theorem that *an integral rational equation of degree n in a single variable x has exactly n roots* (we admit the possibility that some of the roots may be equal). Also, we have seen illustrations of the related fact that, if n is a positive integer, every number H has exactly n distinct nth roots, some or all of which may be imaginary.

EXERCISE 75

Solve by the method of page 196, *without first clearing of fractions when they occur. Results may be left in simplest radical form.*

1. $x^4 - 5x^2 + 4 = 0$. **2.** $x^4 - 10x^2 + 9 = 0$. **3.** $x^4 - 8x^2 + 16 = 0$.

4. $9x^4 + 4 = 13x^2$. **5.** $4x^4 + 15x^2 = 4$. **6.** $y^4 - y^2 = 2$.

7. $y^4 + 7y^2 = 18$. **8.** $x^4 - 9 = 0$. **9.** $81y^4 - 16 = 0$.

10. $x^6 - 8 = 7x^3$. **11.** $27x^6 + 1 = 28x^3$. **12.** $8y^6 + 39y^3 = 5$.

13. $4x^{-4} - 11x^{-2} - 3 = 0$. **14.** $36x^{-4} - 13x^{-2} + 1 = 0$.

15. $25x^{-4} - 26x^{-2} + 1 = 0$. **16.** $2 + 17x^{-2} - 9x^{-4} = 0$.

17. $8x^6 + 35x^3 + 27 = 0$. **18.** $1 - 2x^{-2} - 3x^{-4} = 0$.

19. $(x^2 - x)^2 - (8x^2 - 8x) + 12 = 0$.

20. $(x^2 + 3x)^2 - 3x^2 - 9x - 4 = 0$.

21. $2(2x^2 - x)^2 - 6x^2 + 3x - 9 = 0$.

22. $(x^2 + 4x)^2 - 17x^2 - 60 - 68x = 0$.

23. $\dfrac{2}{(x+1)^2} - \dfrac{7}{x+1} + 6 = 0.$ **24.** $\dfrac{6}{(2x+3)^2} - \dfrac{5}{2x+3} - 4 = 0.$

25. $\dfrac{4}{(x^2+3x)^2} + \dfrac{3}{x^2+3x} = 1.$ **26.** $\dfrac{x^2}{(x+2)^2} + \dfrac{2x}{x+2} = 8.$

27. $\left(x - \dfrac{3}{x}\right)^2 - \left(x - \dfrac{3}{x}\right) = 6.$ **28.** $\left(x - \dfrac{5}{x}\right)^2 - 2x + \dfrac{10}{x} = 24.$

29. $x^4 + 2x^3 + x^2 - (14x^2 + 14x) + 24 = 0.$

30. $4x^4 - 4x^3 + x^2 + 4x^2 - 2x - 15 = 0.$

31. $x^{10} - 30x^5 = 64.$ **32.** $6x^6 + 7x^3 = 20.$

33. $\dfrac{2-x}{x^2} - \dfrac{3x^2}{2-x} - 2 = 0.$ **34.** $\dfrac{y^2+2}{5-2y} = \dfrac{5-2y}{y^2+2} - \dfrac{35}{6}.$

HINT for Problem 33. Let $y = (2-x)/x^2$.

35. $2x^4 - 11ax^2 + 12a^2 = 0.$ **36.** $(2x^2 - 3ax)^2 - 2a^2x^2 + 3a^3x = 2a^4.$

★*Find all roots by first using factoring.*

37. $27x^3 - 8 = 0.$ **38.** $x^3 + 8 = 0.$ **39.** $x^3 - 27 = 0.$ **40.** $16x^4 = 81.$

41. $81 - 625x^4 = 0.$ **42.** $8y^3 - 125 = 0.$ **43.** $125x^3 + 27 = 0.$

★*Find the three cube roots of each number.*

44. $-27.$ **45.** $64.$ **46.** $1.$ **47.** $-1.$ **48.** $8.$ **49.** $\frac{1}{8}.$ **50.** $\frac{1}{27}.$

★*Find the four 4th roots of each number.*

51. 1. **52.** 16. **53.** 81. **54.** 625. **55.** 16. **56.** 256. **57.** $\frac{16}{81}.$

140. An operation sometimes leading to extraneous solutions

Let $M = N$ represent any equation. On squaring both sides, we obtain $M^2 = N^2$, which is satisfied if $M = N$ or if $M = -N$. Hence, the solutions of $M^2 = N^2$ consist of all solutions of $M = N$ together with those of $M = -N$.

ILLUSTRATION 1. $x = 5$ is the only root of $\qquad x - 3 = 2. \qquad (1)$
On squaring both sides, we obtain $\qquad (x-3)^2 = 4. \qquad (2)$
On solving (2) for x we find $x - 3 = \pm 2$; hence, $x = 5$ or $x = 1$.
Therefore, (2) has the root $x = 1$ besides the root $x = 5$ of (1).

If an operation on an equation in x produces a new equation which is satisfied by values of x which are not roots of the given equation, we have agreed to call such values **extraneous roots.** From

the preceding discussion, we observe that, if both members of an equation are *squared,** extraneous roots *may* be introduced.

ILLUSTRATION 2. In Illustration 1, $x = 1$ is an extraneous root.

141. Irrational equations

An *irrational equation* is one in which the variables occur under radical signs or in expressions with fractional exponents.

EXAMPLE 1. Solve for x in the following equations (a) and (b).

(a) $2x - 2 = \sqrt{2x^2 + 4}$.	(b) $2x - 2 = -\sqrt{2x^2 + 4}$.
SOLUTION. 1. Square both sides: 2. $4x^2 - 8x + 4 = 2x^2 + 4$. 3. $2x^2 - 8x = 0$; $2x(x - 4) = 0$. 4. $x = 0$ and $x = 4$. TEST. Substitute $x = 0$ in (a): Does $0 - 2 = \sqrt{4}$? Or, does $-2 = 2$? No. Substitute $x = 4$ in (a): Does $8 - 2 = \sqrt{36}$? Yes. $x = 0$ is *not*, and $x = 4$ *is* a root.	SOLUTION. 1. Square both sides: 2. $4x^2 - 8x + 4 = 2x^2 + 4$. 3. $2x^2 - 8x = 0$; $2x(x - 4) = 0$. 4. $x = 0$ and $x = 4$. TEST. Substitute $x = 0$ in (b): Does $0 - 2 = -\sqrt{4}$? Yes. Substitute $x = 4$ in (b): Does $8 - 2 = -\sqrt{36}$? Or, does $6 = -6$? No. $x = 4$ is *not*, and $x = 0$ *is* a root.

Comment. We met the extraneous roots $x = 0$ in solving (a) and $x = 4$ in solving (b). The test of the values obtained in Step 4 in either solution was necessary in order to *reject* these extraneous roots. The necessity for the test is also shown by the fact that, although (a) and (b) are *different equations*, all distinction between them is lost after squaring.

SUMMARY. *To solve an equation involving radicals:*

1. *Transpose the most complicated radical to one member and all other terms to the other side.*

2. *If the most complicated radical is a square root, square both members; if a cube root, cube both members; etc.*

3. *Repeat Steps* 1 *and* 2 *with the effort to eliminate all radicals involving the unknowns. Then, solve the resulting equation.*

4. *Test each value obtained in Step* 3 *by substitution in the given equation to determine which values are roots.*

* Also true if both sides are raised to any positive integral power.

Note 1. Recall that, if A is positive, $\sqrt{A}$, or $A^{\frac{1}{2}}$, represents the *positive* square root of A and that $\sqrt[n]{A}$ represents only the *principal* nth root of A. Also, in testing for extraneous roots, remember that we are using $a^{\frac{m}{n}}$ to represent only the *principal* nth root of a^m.

EXAMPLE 2. Solve: $(x-2)^{\frac{1}{2}} - \sqrt{2x+5} = 3.$

SOLUTION. 1. $\sqrt{x-2} = 3 + \sqrt{2x+5}.$

2. Square: $x - 2 = 9 + 6\sqrt{2x+5} + 2x + 5.$

3. Simplify: $-x - 16 = 6\sqrt{2x+5}.$

4. Square: $x^2 + 32x + 256 = 36(2x+5);$

$$x^2 - 40x + 76 = 0; \quad (x-38)(x-2) = 0.$$

Possible roots of the given equation are $x = 38$ and $x = 2$.

TEST. Substitute $x = 2$ and $x = 38$ in the original equation:

$x = 2$: does $\sqrt{2-2} - \sqrt{4+5} = 3$, or does $-3 = 3$? No.

$x = 38$: does $\sqrt{38-2} - \sqrt{76+5} = 3$, or does $6 - 9 = 3$? No.

Hence, neither $x = 2$ nor $x = 38$ is a root. Therefore there are *no solutions* for the given equation.

EXERCISE 76

Solve for x or y or z.

1. $\sqrt{x+2} = 3.$
2. $\sqrt{3-x} = 5.$
3. $\sqrt{2-7x} = -4.$
4. $\sqrt{4+2x} = -3.$
5. $\sqrt[3]{3+2x} = 3.$
6. $\sqrt[3]{6x-2} = 4.$
7. $\sqrt[4]{3-x} = 2.$
8. $\sqrt[4]{x+5} = 1.$
9. $(2+x)^{\frac{1}{2}} = 4.$
10. $(3+x)^{\frac{1}{3}} = -2.$
11. $3x = 5\sqrt{2}.$
12. $(z-2)^{\frac{1}{2}} = -2.$
13. $(2x+3)^{\frac{1}{3}} = -5.$
14. $2\sqrt{3} + 5z = 0.$
15. $\sqrt{y} = 6 - y.$
16. $5\sqrt{x} = 3 - 2x.$
17. $3\sqrt{x} + 9 = 2x.$
18. $4x^2 + x\sqrt{3} = 0.$
19. $\sqrt[3]{x^2 - 2x} - 2 = 0.$
20. $\sqrt[4]{x^2 - 24x} - 3 = 0.$
21. $2y^2 - 3y\sqrt{5} = 0.$
22. $\sqrt{2y+5} = 1 + \sqrt{2y}.$
23. $\sqrt{3z+1} = 1 - \sqrt{z}.$
24. $\sqrt{2y+1} + \sqrt{y} = 1.$
25. $\sqrt{2x+4} + \sqrt{2x} = 1.$
26. $\sqrt{x-2} - \sqrt{2x+3} = 2.$
27. $\sqrt{7-2x} - \sqrt{3-x} = 1.$
28. $\sqrt{3-2x} + \sqrt{2+2x} = 3.$
29. $\sqrt{2-4x} + 2\sqrt{1-3x} = 2.$
30. $\sqrt{2x + \sqrt{2x-4}} = 2.$
31. $\sqrt{2\sqrt{x+5} - \sqrt{x}} = 2.$

32. $\sqrt{3+x} - (3-x)^{\frac{1}{2}} = x^{\frac{1}{2}}$. **33.** $2\sqrt{x^2+x-2} - x = x^2 - 2$.

34. Solve for x: $\sqrt{3x+a} - 3\sqrt{x} + \sqrt{a} = 0$.

35. Solve for z: $\sqrt{z-a} + \sqrt{2z+3a} = \sqrt{5a}$.

36. $\sqrt{3+3x} = 2\sqrt{3x-2} - \sqrt{3-x}$.

37. Solve for x: $\sqrt{x} + \sqrt{3x+4b} = 2\sqrt{2x+b}$.

38. Solve $v = \sqrt{2gs}$, (*a*) for s; (*b*) for g.

39. Solve $t = \pi\sqrt{\frac{l}{g}}$, (*a*) for l; (*b*) for g.

★**40** Solve: $4x^{\frac{2}{3}} + 7x^{\frac{1}{3}} - 2 = 0$.

Solution. 1. Let $y = x^{\frac{1}{3}}$; then $y^2 = x^{\frac{2}{3}}$ and $4y^2 + 7y - 2 = 0$.

2. Solve for y: $(4y-1)(y+2) = 0$; $y = -2$ and $y = \frac{1}{4}$.

3. If $y = -2$, then $x^{\frac{1}{3}} = -2$ or $\sqrt[3]{x} = -2$; hence, $x = -8$.

4. If $y = \frac{1}{4}$, then $x^{\frac{1}{3}} = \frac{1}{4}$ or $\sqrt[3]{x} = \frac{1}{4}$; hence, $x = (\frac{1}{4})^3 = \frac{1}{64}$.

★*Solve by reducing to a quadratic in some new variable.*

41. $5z + 3\sqrt{z} = 2$. **42.** $2x^{\frac{1}{2}} + 9x^{\frac{1}{4}} = 5$. **43.** $3x^{-1} + 5 = 8x^{-\frac{1}{2}}$.

44. $3x + 7x^{\frac{1}{2}} = 6$. **45.** $2x^{\frac{2}{3}} = 6 + x^{\frac{1}{3}}$. **46.** $2x^{-1} + x^{-\frac{1}{2}} = 6$.

47. $3x^{\frac{1}{2}} = 8x^{\frac{1}{4}} - 4$. **48.** $4x^{\frac{2}{3}} = 7x^{\frac{1}{3}} + 2$. **49.** $4x^{-1} + 3x^{-\frac{1}{2}} = 1$.

50. $x^2 + 2x - \sqrt{x^2+2x-6} = 12$. **51.** $2x^2 + 3\sqrt{2x^2+3} = 7$.

★*Find all real roots.*

52. $y^{\frac{3}{2}} = 8$. **53.** $x^{\frac{5}{2}} = 32$. **54.** $z^{\frac{2}{3}} = 16$. **55.** $x^{\frac{3}{4}} = -8$.

56. $y^{\frac{2}{3}} = 4$. **57.** $x^{\frac{2}{5}} = 9$. **58.** $z^{\frac{5}{2}} = -243$. **59.** $x^{\frac{3}{5}} = -27$.

60. $(2x+1)^{\frac{2}{3}} = 4$. **61.** $(5-3x)^{\frac{3}{4}} = 27$. **62.** $(2+3x)^{\frac{3}{2}} = 8$.

63. $2x^{-3} + 15x^{-\frac{3}{2}} - 8 = 0$. **64.** $3x^3 + 26x^{\frac{3}{2}} - 9 = 0$.

★142. Miscellaneous problems about the roots

Example 1. If c is a constant and 2 is one root of the equation

$$3x^2 - 7x + c = 0,$$

find the other root.

Solution. Let the roots be r and s, with $r = 2$. Then, from page 193,

$$r + s = \tfrac{7}{3}; \quad \text{or,} \quad 2 + s = \tfrac{7}{3}; \quad s = \tfrac{1}{3}.$$

EXAMPLE 2. Find the constant h if one root of the following equation exceeds the other by 5: $x^2 - x - 2h = 0.$

SOLUTION. 1. Given condition: $r - s = 5.$ (1)

2. Sum of the roots: $r + s = 1.$ (2)

3. Product of the roots: $rs = -2h.$ (3)

4. Solve (1) and (2) for r and s: $r = 3; \quad s = -2.$ (4)

5. Substitute (4) in (3): $-6 = -2h; \quad h = 3.$ (5)

EXAMPLE 3. Find the values of k for which the following equation in x has equal roots:

$$kx^2 + 2x^2 - 3kx + k = 0.$$

SOLUTION. 1. Group the terms in standard form:

$$(k + 2)x^2 - 3kx + k = 0.$$

Hence, the standard coefficients are $a = k + 2$, $b = -3k$, and $c = k$.

2. If the roots are equal, the discriminant $b^2 - 4ac$ is zero:

$$\textit{discriminant} = (-3k)^2 - 4(k + 2)(k) = 0; \quad \text{or} \quad 5k^2 - 8k = 0.$$

3. Hence, $k(5k - 8) = 0$; or $k = 0$ and $k = \frac{8}{5}$.

THEOREM I. *If one root of $ax^2 + bx + c = 0$ is the negative of the other root, then $b = 0$.*

Proof. 1. If r and s are the roots, then $r = -s$, or $r + s = 0$.

2. Hence, $r + s = -\dfrac{b}{a} = 0$, or $-b = 0$. Therefore $b = 0$.

THEOREM II. *If $b = 0$, then one root of $ax^2 + bx + c = 0$ is the negative of the other root.*

Proof. Since $b = 0$, then $-\dfrac{b}{a} = 0 = r + s$. Hence, $r = -s$.

Note 1. Theorem II is the **converse** of Theorem I. Theorem II could have been abbreviated by adding the words *and conversely* at the end of Theorem I. Or, *both* theorems are included in the following statement: "*One root of $ax^2 + bx + c = 0$ is the negative of the other root* **when and only when** $b = 0$." In this statement we justify the phrase *only when* by Theorem I, and the word *when* by Theorem II.

EXAMPLE 4. Find the values of the constant h so that the equation $hx^2 + 9h^2x = 3 + x$ will have one root the negative of the other.

SOLUTION. 1. Write in standard form: $hx^2 + x(9h^2 - 1) - 3 = 0.$

2. From Theorem II: $9h^2 - 1 = 0; \quad h = \pm \frac{1}{3}.$

★EXERCISE 77

By use of the discriminant, find the values of the constant k for which the equation will have equal roots for the unknown x.

1. $4x^2 - 3kx + 1 = 0.$ **2.** $4x^2 + 5kx + 4 = 0.$ **3.** $2kx^2 + 9 = 12x.$

4. $kx^2 + 3kx + 5 = 0.$ **5.** $x^2 - kx^2 - 5kx = 3k.$

6. $5x^2 - 2kx - k = 0.$ **7.** $k^2x^2 - kx - x^2 - x = 3.$

8. $x^2 - kx + x - k = 0.$ **9.** $kx + x^2 + kx^2 - 2x = 4.$

Find the values of the constant k for which the graph of the function of x will be tangent to the x-axis.

10. $5x^2 - 2kx + k.$ **11.** $2kx^2 - 3kx + 5.$ **12.** $x^2 - 3x - k - kx.$

13. $2x^2 + 2x - 3k - 2kx.$ **14.** $2x^2 - 2kx^2 - 5kx + 5.$

In all problems, x is the unknown and all other letters are constants.

15. If one root is 3, find the other root: $2x^2 - 5x + d = 0.$

16. If one root is -2, find the other root: $-3x^2 + dx + 5 = 0.$

17. If one root is 5, find —the other root: $2x^2 + bx - 3 = 0.$

18. If one root is $\frac{1}{4}$, find the other root: $3x^2 + 7x + h = 0.$

Find the value of the constant h under the given condition.

19. The sum of the roots is -5: $3hx^2 - 4x - 5hx + 6 = 0.$

20. The sum of the roots is 7: $5x^2 - hx^2 + 54hx + 4 = 0.$

21. The product of the roots is 9: $2x^2 - 3hx^2 - 6x + 4h = 0.$

22. The product of the roots is -6: $3hx^2 + 5x + h - 1 = 0.$

23. One root exceeds the other by 2: $2x^2 - 4h + 5x = 0.$

24. One root exceeds the other by -3: $3x^2 - 5x + 3h - 6 = 0.$

25. One root is four times the other: $2x^2 + 20x + h^2 = 13.$

One root is the negative of the other; find h.

26. $hx^2 - 6x + 3hx - 5 = 0.$ **27.** $2hx^2 - 3hx - 5h^2x + 6 = 0.$

28. $x^2 + 12x - 3h^2x + h = 0.$ **29.** $h^3x^2 + 3h^2x + 5hx - 4 = 0.$

30. $hx^2 + 9h^2x = 3 + x.$ **31.** $x^2 - 3h^2x = h - 2x.$

32. $3x^2 + 5h^2x = 2 + x.$ **33.** $x^2 - hx - h^2x + 2x = 0.$

34. Prove that, if $ax^2 + bx + c = 0$ has one root zero, then $c = 0$, and conversely.

CHAPTER 12

THE BINOMIAL THEOREM

143. Expansion of a positive integral power of a binomial

By multiplication, we obtain the following results:

$$(x + y)^1 = x + y;$$
$$(x + y)^2 = x^2 + 2xy + y^2;$$
$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3;$$
$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4;$$
$$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5.$$

We see that, if $n = 1, 2, 3, 4$, or 5, *the expansion of* $(x + y)^n$ *contains* $(n + 1)$ **terms** *with the following properties:*

I. *In any term the sum of the exponents of x and y is n.*

II. *The first term is x^n, and in each other term the exponent of x is 1 less than in the preceding term.*

III. *The second term is $nx^{n-1}y$, and in each succeeding term the exponent of y is 1 more than in the preceding term.*

IV. *If the coefficient of any term is multiplied by the exponent of x in that term and if the product is divided by the number of that term, the quotient obtained is the coefficient of the next term.*

ILLUSTRATION 1. In $(x + y)^4$, the third term is $6x^2y^2$. By Property IV, we obtain $(6 \cdot 2) \div 3$, or 4, as the coefficient of the fourth term. In $(x + y)^5$, the fourth term is $10x^2y^3$; by Property IV, we obtain $(10 \cdot 2) \div 4$, or 5, as the coefficient of the fifth term.

V. *The coefficients of terms equidistant from the ends are the same.*

ILLUSTRATION 2. The coefficient of the second term equals that of the next to the last term, etc.

We shall assume that Properties I to V are true if n is any positive integer, although *we have merely verified their truth when* $n = 1, 2, 3, 4,$ *and* 5. The theorem which justifies this assumption is called the **binomial theorem,** which we shall accept without proof in this text.

Example 1. Expand $(c + w)^7$.

Solution. 1. By use of Properties I, II, and III, we obtain

$$(c + w)^7 = c^7 + 7c^6w + \quad c^5w^2 + \quad c^4w^3 + \quad c^3w^4 + \quad c^2w^5 + \quad cw^6 + w^7,$$

where spaces are left for the unknown coefficients.

2. By Property IV, the coefficient of the third term is $(7 \cdot 6) \div 2$, or 21; that of the fourth term is $(21 \cdot 5) \div 3$, or 35.

3. By Property V, we obtain the other coefficients; hence,

$$(c + w)^7 = c^7 + 7c^6w + 21c^5w^2 + 35c^4w^3 + 35c^3w^4 + 21c^2w^5 + 7cw^6 + w^7.$$

Example 2. Expand $\left(2a - \frac{w}{3}\right)^6$.

Solution. 1. $\left(2a - \frac{w}{3}\right)^6 = \left[(2a) + \left(-\frac{w}{3}\right)\right]^6$.

2. We use Properties I to V with $x = 2a$ and $y = -\frac{w}{3}$, and keep the terms of the binomial within parentheses in finding the coefficients:

$$\left(2a - \frac{w}{3}\right)^6 = (2a)^6 + 6(2a)^5\left(-\frac{w}{3}\right) + 15(2a)^4\left(-\frac{w}{3}\right)^2 + 20(2a)^3\left(-\frac{w}{3}\right)^3$$

$$+ 15(2a)^2\left(-\frac{w}{3}\right)^4 + 6(2a)\left(-\frac{w}{3}\right)^5 + \left(-\frac{w}{3}\right)^6, \quad \text{or}$$

$$\left(2a - \frac{w}{3}\right)^6 = 64a^6 - 64a^5w + \frac{80}{3}a^4w^2 - \frac{160}{27}a^3w^3 + \frac{20}{27}a^2w^4 - \frac{4}{81}aw^5 + \frac{w^6}{729}.$$

Note 1. The following array of numbers is called *Pascal's Triangle.* The successive rows give the coefficients in the successive positive integral powers of $x + y$. To form any row after the second, we first place 1 at the left; the 2d number is the sum of the 1st and 2d numbers in the preceding row; the 3d number in the new row is the sum of the 2d and 3d numbers in the preceding row; etc. This triangle was known to Chinese mathematicians in the early fourteenth century, and it appeared in print in Europe for the first time in 1527.

					1					
				1		1				
			1		2		1			
		1		3		3		1		
	1		4		6		4		1	
1		5		10		10		5		1

The preceding diagram exhibits the fact that the *largest* coefficient in any power of $x + y$ is the coefficient of the *central term* or *terms*.

We observe that the signs are *alternately plus and minus* in the expansion of a power of a binomial where one term bears a *plus* sign and the other term bears a *minus* sign.

144. The factorial symbol

The symbol $n!$ is read "*n factorial*," and is an abbreviation for *the product of all integers from* 1 *to n inclusive*, where n is a positive integer.

ILLUSTRATION 1. $5! = 1\cdot2\cdot3\cdot4\cdot5 = 120.$ $3! = 1\cdot2\cdot3 = 6.$

$$\frac{7!}{10!} = \frac{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9\cdot10} = \frac{1}{8\cdot9\cdot10} = \frac{1}{720}.$$

EXERCISE 78

Expand each power by use of Properties I *to* V.

1. $(a + b)^5$.
2. $(c - d)^6$.
3. $(x - y)^8$.
4. $(c + 3)^5$.
5. $(2 + a)^4$.
6. $(x - 2a)^7$.
7. $(3b - y)^6$.
8. $(2c + 3d)^3$.
9. $(a + b^2)^3$.
10. $(c^3 - 3d)^4$.
11. $(a^2 - b^2)^6$.
12. $(c - x^3)^5$.
13. $(x - \frac{1}{2})^5$.
14. $(1 - a)^9$.
15. $(\sqrt{x} - \sqrt{y})^6$.
16. $(x^{\frac{1}{2}} + a)^5$.
17. $(- a + y^{-2})^4$.
18. $(z^{-3} - x)^5$.
19. $(x^{\frac{1}{2}} - 2a^{-1})^4$.
20. $\left(\frac{1}{x} + \frac{2}{y}\right)^3$.
21. $\left(\frac{2}{a} - 3b^2\right)^4$.
22. $\left(\frac{y^2}{x} + \frac{b^{\frac{1}{2}}}{y}\right)^5$.

Find only the first three terms of the expansion.

23. $(a + 12)^{15}$.
24. $(c - 3)^{25}$.
25. $(a^2 + b^3)^{20}$.
26. $(1 + 2a)^{10}$.
27. $(1 - .1)^{22}$.
28. $(1 + .2)^{12}$.
29. $(1 - \sqrt{2})^{12}$.
30. $(1 - 3x^3)^{18}$.
31. $(2x - a^2)^{30}$.
32. $(x^{\frac{1}{2}} + b)^{14}$.
33. $(a^{-1} + 3)^{26}$.
34. $(x - a^{-2})^{11}$.
35. $(x - y)^n$.
36. $(a + x)^k$.
37. $(x^2 - y)^m$.
38. $(w^2 + z)^h$.

Compute each factorial expression.

39. $6!$
40. $8!$
41. $11!$
42. $\frac{7!}{3!}$
43. $\frac{9!}{5!\,4!}$

145. General term of the binomial expansion

By use of Properties I to IV of Section 143, we obtain

$$\left.\begin{aligned}(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{1\cdot 2}x^{n-2}y^2 \\ + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^{n-3}y^3 + \cdots.\end{aligned}\right\} \quad (1)$$

In (1) we read the dots "$\cdots$" as "*and so forth.*" In the terms in y, y^2, and y^3, we observe special cases of the following facts, which we shall accept without proof.

SUMMARY. *Description of the term involving y^r, in $(x+y)^n$:*

A. *The exponent of x is $n - r$.*

B. *The denominator is the product of all integers from* 1 *to r inclusive; that is, the denominator is $r!$.*

C. *The numerator of the coefficient has r factors, the first being n and each other being* 1 *less than the preceding factor. The last factor is $n - r + 1$.*

When (A), (B), and (C) are combined, they state that

$$\textit{the term involving } y^r \textit{ is } \frac{n(n-1)\cdots(n-r+1)}{r!}x^{n-r}y^r. \quad (2)$$

By use of formula 2, we may write

$$\left.\begin{aligned}(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \cdots \\ + \frac{n(n-1)\cdots(n-r+1)}{r!}x^{n-r}y^r + \cdots + y^n.\end{aligned}\right\} \quad (3)$$

We refer to (3) as the **binomial formula.** By use of (2), we can write any term of (3) without writing the other terms. Hence, we refer to (2) as the *general* term of the expansion of $(x+y)^n$.

ILLUSTRATION 1. The term involving y^4 in the expansion of $(x+y)^7$ is

$$\frac{7\cdot 6\cdot 5\cdot 4}{4!}x^3y^4 \quad \text{or} \quad 35x^3y^4.$$

EXAMPLE 1. Find the 8th term of $(3a^{\frac{1}{2}} - b)^{11}$.

SOLUTION. The 8th term will involve the 7th power of the 2d term of the binomial. Hence, use (2) with $r = 7$, $x = 3a^{\frac{1}{2}}$, and $y = -b$:

$$\text{8th term is } \frac{11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}(3a^{\frac{1}{2}})^4(-b)^7 = -26{,}730a^2b^7.$$

Note 1. To derive a formula for the rth term in (3), we notice that this term will contain y^{r-1} as a factor. Hence, we substitute $(r - 1)$ for r in (2) and find that

$$\textit{the } r\text{th } \textit{term is } \frac{n(n-1)\cdots(n-r+2)}{(r-1)!}x^{n-r+1}y^{r-1}. \tag{4}$$

We may call (4), as well as (2), the *general term.* Example 1 could have been solved by use of (4) with $r = 8$.

EXAMPLE 2. Find the term involving z^{12} in the expansion of $(v - z^3)^7$.

SOLUTION. Since $z^{12} = (z^3)^4$, we use formula 2 with $n = 7$, $r = 4$, $x = v$, and $y = -z^3$: the term is $\dfrac{7\cdot6\cdot5\cdot4}{1\cdot2\cdot3\cdot4}v^3(-z^3)^4$, or $35v^3z^{12}$.

EXAMPLE 3. Compute $(1.01)^6$ correct to 3 decimal places.

SOLUTION. $(1.01)^6 = (1 + .01)^6$

$= 1^6 + 6(1)^5(.01) + 15(1)^4(.01)^2 + \cdots$

$= 1 + 6(.01) + 15(.01)^2 + 20(.01)^3 + \cdots$

$= 1 + .06 + .0015 + .000020 + \cdots$ (negligible terms)

$= 1.06152 = 1.062$, approximately.

EXERCISE 79

Find only the specified term.

1. Term involving y^5 in the expansion of $(a + y)^9$.
2. Term involving x^6 in the expansion of $(z + x)^{10}$.
3. Term involving y^4 in the expansion of $(x - y)^7$.
4. Term involving x^5 in the expansion of $(x + 3y)^8$.
5. 4th term of $(a - x)^9$.
6. 3d term of $(w - z)^{11}$.
7. 6th term of $(a^2 + x)^7$.
8. 10th term of $(x^2 + y^3)^{10}$.
9. 4th term of $(x - 5y)^7$.
10. 5th term of $(1 - .02)^7$.
11. 6th term of $(1 + .1)^8$.
12. 4th term of $(\frac{1}{2}x - 2y^2)^9$.
13. Term involving z^8 in the expansion of $(x - z^2)^6$.
14. Term involving w^{10} in the expansion of $(w^2 + y^3)^8$.
15. Term involving y^6 in the expansion of $(x - y)^n$.

16. Term involving x^3 in the expansion of $(a - x^{\frac{1}{2}})^{10}$.

17. Term involving $x^{\frac{5}{3}}$ in the expansion of $(y + 2x^{\frac{1}{3}})^7$.

18. Middle term of $(x^3 - y)^{10}$.

19. Middle term of $(a + 3x^2)^8$.

20. Middle terms of $(a^{\frac{1}{3}} + y)^7$.

21. Middle terms of $(2x^{\frac{1}{2}} + y^2)^9$.

22. Term involving $\dfrac{1}{x^4}$ in the expansion of $\left(\dfrac{a}{b} - \dfrac{1}{x}\right)^6$.

23. Term involving $\dfrac{a^3}{x^3}$ in the expansion of $\left(z^3 + \dfrac{3a}{x}\right)^7$.

24. Term with x^4 as a factor in the expansion of $\left(\dfrac{y}{x} - \dfrac{x}{2}\right)^8$.

Find the term or terms with the largest coefficient in the expansion of the power.

25. $(a + x)^8$.
26. $(c - w)^{10}$.
27. $(a^2 + b)^9$.
28. $(c - d^3)^{11}$.

Compute by use of the binomial theorem. In any problem involving decimals, use only enough terms to obtain the result correct to three decimal places.

29. $(10 - a)^4$.
30. $(100 - 2)^3$.
31. 99^4.
32. 39^4.
33. 51^3.
34. $(1.01)^7$.
35. $(1.01)^{12}$.
36. $(1.02)^8$.
37. $(1.03)^7$.
38. $(.99)^6$.
39. $(.98)^5$.
40. $(1.04)^{10}$.
41. $(1.02)^{11}$.
42. $(.52)^8$.
43. $(.49)^9$.
44. 101^5.
45. 62^4.

CHAPTER 13

RATIO, PROPORTION, AND VARIATION

146. Ratio

The *ratio* of one number a to a second number b is the quotient a/b. The ratio of a to b is sometimes written $a:b$. A ratio is a fraction, and any fraction can be described as a ratio:

$$a:b = \frac{a}{b}. \tag{1}$$

The ratio of two concrete quantities has meaning only if they are of the same kind. Their ratio is the quotient of their measures in terms of the same unit.

ILLUSTRATION 1. The ratio of 3 feet to 5 inches is $\frac{36}{5}$.

147. Proportion

A *proportion* is a statement that two ratios are equal. That is, a proportion is merely a statement that two fractions are equal. The proportion

$$a:b = c:d \quad \textit{means that} \quad \frac{a}{b} = \frac{c}{d}. \tag{1}$$

The proportion $a:b = c:d$ is read "a is to b as c is to d." We say that the four numbers a, b, c, and d *form a proportion.*

In a proportion $a:b = c:d$, the first and fourth numbers, a and d, are called the **extremes,** and the second and third, b and c, are called the **means** of the proportion.

ILLUSTRATION 1. To solve the proportion $x:(25 - x) = 3:7$, we first change it to fractional form, and then solve the resulting equation:

$$\frac{x}{25 - x} = \frac{3}{7}; \quad 7x = 75 - 3x; \quad 10x = 75; \quad \text{hence,} \quad x = 7.5.$$

EXAMPLE 1. Divide 36 into two parts with the ratio 3:7.

SOLUTION. 1. Let x and y be the parts; then $$x + y = 36. \qquad (2)$$

2. Also, $x:y = 3:7$, or $\frac{x}{y} = \frac{3}{7}\cdot$ Hence, $$7x = 3y. \qquad (3)$$

3. On solving the system [(2), (3)] we obtain ($x = 10.8$, $y = 25.2$).

Note 1. If two triangles (or polygons of any number of sides) are similar, then (*a*) *the ratio of any two sides of one triangle equals the ratio of the corresponding sides of the other triangle,* and (*b*) *the area of one triangle is to the area of the other as the square of any side of the first triangle is to the square of the corresponding side of the other triangle.*

EXAMPLE 2. The sides of a triangle are 12, 8, and 15 inches long. In a similar triangle, the longest side is 40 inches long. Find the other sides.

SOLUTION. 1. Let x and y be the lengths in inches of the sides of the similar triangle corresponding to those sides which are 8 and 12 inches long in the first triangle. Then,

$$y:12 = 40:15 \quad \text{or} \quad \frac{y}{12} = \frac{40}{15}; \qquad (4)$$

$$x:8 = 40:15 \quad \text{or} \quad \frac{x}{8} = \frac{40}{15}. \qquad (5)$$

2. Solving (4) and (5) we find $y = 32$ feet and $x = 21\frac{1}{3}$ feet.

EXERCISE 80

Express each ratio as a fraction and simplify.

1. $\frac{3}{8}:\frac{5}{16}$. 2. $\frac{14}{3}:\frac{7}{4}$. 3. $5\frac{1}{2}:7\frac{1}{3}$. 4. $x^3y^4:x^5y^3$. 5. $az^2:a^4z$.

Find the ratio of the given quantities.

6. 75 pounds to 160 ounces.
7. 27 days to 156 hours.
8. 51 pints to 17 quarts.
9. 25 miles to 3175 yards.
10. 72 cubic feet to 1320 cubic inches.

Change to fractional form and solve.

11. $3:(20 - 2x) = 5:2$.
12. $(2 - 3y):(4 + 5y) = 3:2$.
13. $x:(x - 25) = 6$.
14. $(2 - x):(3 + x) = (4 - x):(2 + x)$.
15. $2x:(5 - 3x) = 2$.
16. $(4 + x):(3 + x) = -2:5x$.

17. A line 18 inches long is divided into two parts whose lengths have the ratio 5:4. Find the lengths.

Solve by introducing one or more unknowns.

18. Divide 45 into two parts whose ratio is 4:11.

19. Divide 90 into two parts such that the ratio of one part decreased by 5 to the other part decreased by 10 is 1:4.

20. Find two numbers whose difference is 18 and whose ratio is 4:3.

21. The sides of a triangle are 12, 8, and 18 inches long. In a similar triangle, the shortest side is 40 inches long. Find the other sides.

22. The sides of a polygon are 10, 7, 4, and 8 inches long. If the longest side is lengthened by 2 feet, by how much should the other sides be lengthened to obtain a similar polygon?

23. A triangle whose base is 15 inches long has an area of 220 square inches. Find the area of a similar triangle whose base is $6\frac{1}{4}$ feet long.

24. The area of a quadrilateral is 49 square feet and its longest side is 12 feet long. Find the area of a similar quadrilateral whose longest side is 15 feet long.

25. The area of a triangle is 150 square feet and its shortest side is 12 feet long. Find the shortest side of a similar triangle whose area is 30 square feet.

26. A man 6 feet tall stands at the foot of a tower and casts a shadow 10 feet long. How high is the tower if its shadow is 69 feet long?

27. A man $5\frac{1}{2}$ feet tall stands 40 feet from a street light and casts a shadow $9\frac{1}{2}$ feet long. How high is the light?

28. Solve the proportion $3:x = x:27$ for x.

29. Solve the proportion, $a:x = x:b$ for x.

If $a:x = x:b$, then x is called a **mean proportional** *between a and b. If $a:x = x:b$, then $x^2 = ab$ or $x = \pm\sqrt{ab}$; or, if neither a nor b is zero, there are two mean proportionals between a and b. Find the mean proportionals between each of the following pairs of numbers.*

30. 64 and 4. **31.** -4 and $-\frac{1}{4}$. **32.** 2 and 8.

33. 25 and 25. **34.** -2 and 8. **35.** -3 and 27.

36. $2a^3$ and $4a$. **37.** y^2 and x^{-4}. **38.** x^4y^2 and x^4y^6.

39. $\dfrac{y^2+y-6}{4-y}$ and $\dfrac{y^2-y-12}{2-y}$. **40.** (z^3-8) and $\dfrac{z^2+2z+4}{z-2}$.

★*If $a:b = c:x$, then x is called the* **fourth proportional** *to a, b, and c. Find the fourth proportional to each set of numbers:*

41. 2, -5, and 14. **42.** 5, 4, and 7. **43.** 3, b, and a^2b.

★*If* $a:b = b:x$, *then* x *is called the* **third proportional** *to* a *and* b. *Find the third proportional to each pair of numbers:*

44. 18; 50. **45.** $2\frac{1}{3}$; $\frac{3}{5}$. **46.** $2xy$; y. **47.** $5m^2n$; $3m^3$.

★*If* $a:b = c:d$ *and if no denominator involved is zero, prove the following properties of the proportion.*

48. PROPERTY I. $ad = bc$; or, in any proportion, *the product of the means equals the product of the extremes.*

49. PROPERTY II. $\frac{a}{c} = \frac{b}{d}$; or, *the means may be interchanged without destroying the proportion.* (The resulting proportion is said to be obtained from $a:b = c:d$ by *alternation.*)

50. PROPERTY III. $\frac{b}{a} = \frac{d}{c}\cdot$ (The resulting proportion is said to be obtained from $a:b = c:d$ by *inversion.*)

51. PROPERTY IV. $\frac{a+b}{b} = \frac{c+d}{d}\cdot$ (Said to be obtained by *composition.* To prove, add 1 to both sides of $a:b = c:d$.)

52. PROPERTY V. $\frac{a-b}{b} = \frac{c-d}{d}\cdot$ (Said to be obtained by *division.*)

148. Direct variation

Let x and y be related variables. Then, we say that

y is proportional to x, or
y varies directly as x, or
y is directly proportional to x, or
y varies as x,

in case there exists a constant k such that, for every value of x, there is a corresponding value of y given by

$$y = kx. \tag{1}$$

We call k the **constant of proportionality** or the **constant of variation.**

ILLUSTRATION 1. The circumference C of a circle varies directly as the radius r because $C = 2\pi r$. The constant of proportionality is 2π.

From $y = kx$, we obtain $k = \frac{y}{x}\cdot$ Hence, if y is proportional to x, the ratio of corresponding values of y and x is a constant. Con-

versely, if the ratio of corresponding values of two variables y and x is a constant, then y is proportional to x, because the equation $\frac{y}{x} = k$ leads to $y = kx$.

ILLUSTRATION 2. If y is a function of x and if it is known that $\frac{y}{x} = 4$, then $y = 4x$, and y is proportional to x.

If y is proportional to x, then x is proportional to y. In other words, the proportionality relationship is a *reciprocal* property. This is true because, if $y = kx$, then

$$x = \frac{1}{k}y. \tag{2}$$

Hence, if y varies as x, with k as the constant of proportionality, then x varies as y, with $1/k$ as the constant of proportionality.

If y varies directly as x, so that equation 1 is true, then the graph of the relationship is a straight line, because (1) is *linear* in x and y. We observe that, for any value of k, the graph of (1) passes through the origin, because $(x = 0,\ y = 0)$ is a solution of (1).

ILLUSTRATION 3. If y is proportional to x, with 3 as the constant of proportionality, then $y = 3x$. The graph of this equation is given in Figure 16.

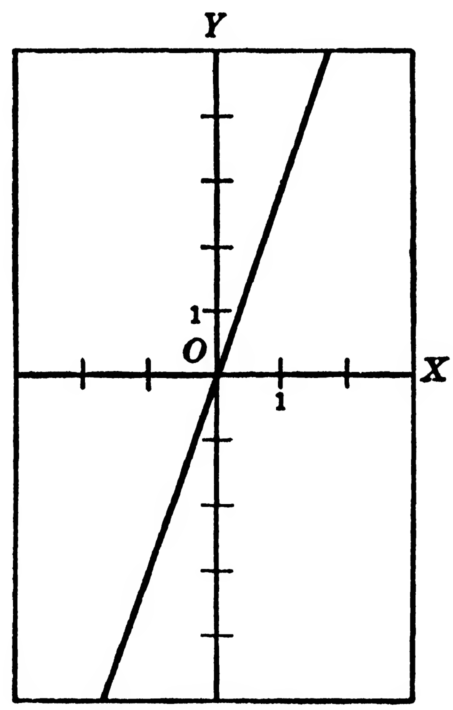

Fig. 16

149. Inverse variation

We say that

y is inversely proportional to x, or
y varies inversely as x,

in case there exists a constant k such that, for every value of x, there is a corresponding value of y given by

$$y = \frac{k}{x}. \tag{1}$$

From this equation, $k = xy$, or the *product* of corresponding values of x and y is a constant.

ILLUSTRATION 1. The time t necessary for a train to go a given distance d varies inversely as the speed s of the train because $t = d/s$. The constant of proportionality here is d.

If y varies inversely as x, with k as the constant of proportionality, then likewise x varies inversely as y, because the equation $k = xy$, which comes from (1), leads to both of the equations

$$y = \frac{k}{x} \quad \text{and} \quad x = \frac{k}{y}. \qquad (2)$$

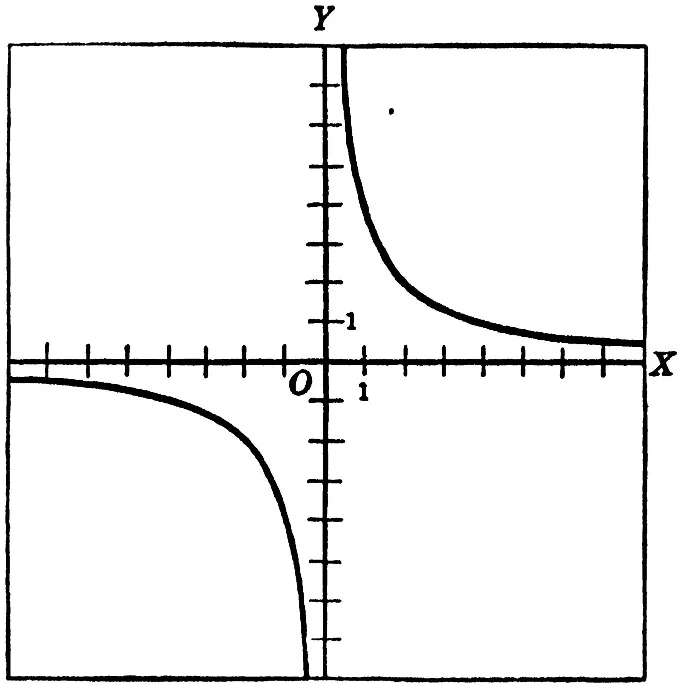

Fig. 17

ILLUSTRATION 2. If y varies inversely as x, with 4 as the constant of proportionality, then $y = 4/x$ or $xy = 4$. The graph of y as a function of x is the graph of the equation $xy = 4$. This graph has *no points* for which $x = 0$ or $y = 0$ because in such cases $xy = 0$, and hence $xy \neq 4$. We make up the following table of values by substituting the values of x in $y = 4/x$. The graph, in Figure 17, extends beyond all limits upward and downward, approaching the y-axis as shown. Similarly, as x grows numerically large without bound, either through positive or through negative values, y approaches zero and the curve approaches the x-axis. The curve in Figure 17 is an illustration of a **hyperbola.***

x	-8	-4	-2	-1	$-\frac{1}{4}$	0	$\frac{1}{4}$	1	2	4	8
y	$-\frac{1}{2}$	-1	-2	-4	-16	★	16	4	2	1	$\frac{1}{2}$

150. Joint variation

We say that

$$\left.\begin{array}{l} z \textit{ varies jointly as } x \textit{ and } y, \text{ or} \\ z \textit{ is directly proportional to } x \textit{ and } y, \text{ or} \\ z \textit{ is proportional to } x \textit{ and } y, \text{ or} \\ z \textit{ varies as } x \textit{ and } y, \end{array}\right\} \qquad (1)$$

in case z is proportional to the product xy, or

$$z = kxy, \qquad (2)$$

* Graphs of hyperbolas are considered in detail in Chapter 16.

where k is a constant of proportionality. Notice that the significance of the word *and* in each statement in (1) is that x and y are *multiplied* in (2). Any of the various types of variation may be combined.

ILLUSTRATION 1. To say that z varies *directly* as x and y and *inversely* as w^3 means that $z = kxy/w^3$.

ILLUSTRATION 2. If $P = 10x^2y/z^3$, then P varies directly as x^2 and y, and inversely as z^3.

151. Applications of variation equations

Suppose it is known that certain variables are related by a variation equation, with an unknown constant of proportionality, k. Then, if one set of corresponding values of the variables is given, we can find k by substituting the values in the variation equation.

EXAMPLE 1. If y is proportional to x and w^2, and if $y = 36$ when $x = 2$ and $w = 3$, find y when $x = 3$ and $w = 4$.

SOLUTION. 1. We are given that $y = kw^2x$, where k is an unknown constant.

2. To find k, substitute ($y = 36$, $x = 2$, $w = 3$) in $y = kw^2x$:

$$36 = k(3^2)(2); \quad 36 = 18k \quad \text{or} \quad k = 2. \tag{1}$$

3. From (1),
$$y = 2w^2x. \tag{2}$$

4. Substitute ($x = 3$, $w = 4$) in (2): $\quad y = 2 \cdot 16 \cdot 3 = 96.$

Notice that the following steps were taken in Example 1.

1. *The variation statement was translated into an equation involving an unknown constant of proportionality.*
2. *The unknown constant was found by substituting given data.*
3. *The value of the constant of proportionality was substituted in the equation of variation, and this equation was used to obtain the value of one variable by use of given values of the other variables.*

Useful information can be obtained by use of an equation of variation on many occasions when the data are not sufficient to enable us to find the value of the constant of proportionality.

EXAMPLE 2. The kinetic energy of a moving body is proportional to the square of its velocity. Find the ratio of the kinetic energy of an automobile traveling at 50 miles per hour to the kinetic energy of the same automobile traveling at 20 miles per hour.

SOLUTION. 1. Let E be the energy, and v the velocity in miles per hour. Then, $E = kv^2$, where k is a constant of proportionality. (The data do not permit us to find the value of k.)

2. Let E_1 be the energy at 20 miles per hour, and E_2 the energy at 50 miles per hour. Then,

$$E_1 = k(20)^2 \quad \text{or} \quad E_1 = 400k; \tag{3}$$

$$E_2 = k(50)^2 \quad \text{or} \quad E_2 = 2500k. \tag{4}$$

3. From (3) and (4),

$$\frac{E_2}{E_1} = \frac{2500k}{400k} = \frac{25}{4} = 6\tfrac{1}{4}.$$

Thus, E_2 is $6\frac{1}{4}$ times as large as E_1.

Note 1. In applications of an equation of variation, the constant of proportionality will depend on the *units* in terms of which the variables in the problem are measured. Hence, if the constant is determined for one set of units, care must be exercised to employ the same units whenever this value of the constant is used.

EXERCISE 81

Introduce letters if necessary and express the relation by an equation involving an unknown constant of proportionality.

1. H varies directly as x and inversely as w^2.

2. B is proportional to x^2 and inversely proportional to z.

3. Z is proportional to $\sqrt{x}$ and varies inversely as y^2.

4. K is proportional to z and w^2 and inversely proportional to xy.

5. $(x + 2)$ is inversely proportional to $(y + 3)$.

6. The area of a triangle is proportional to its altitude.

7. The volume of a sphere is proportional to the cube of its radius.

8. The volume of a specified quantity of gas varies inversely as the pressure applied to it, if the temperature remains unchanged.

9. The weight of a body above the surface of the earth varies inversely as the square of the distance of the body from the earth's center.

10. The power available in a jet of water varies jointly as the weight of the water per cubic foot, the cube of the water's velocity, and the cross-section area of the jet.

11. The maximum horsepower of the boiler which can be served by a chimney of given cross-section area is proportional to the square root of the height of the chimney.

For each formula, give a statement about the variable on the left side in the language of variation. All letters except the constant k represent variables.

12. $y = 7w.$ **13.** $z = -3x^2.$ **14.** $z = 5xy^2.$ **15.** $u = 7x^3y.$

16. $y = \dfrac{kx}{z^3}.$ **17.** $w = \dfrac{3\sqrt{x}}{y^2}.$ **18.** $u = \dfrac{kxy^2}{z}.$ **19.** $w = \dfrac{k\sqrt{x}}{yz^2}.$

By employing all data, obtain an equation relating the variables.

20. P is directly proportional to x^2 and $P = 18$ if $x = 4$.

21. R is inversely proportional to x and directly proportional to y, while $R = 4$ when $x = 3$ and $y = 5$.

22. U varies directly as x and y, and inversely as z^2; $U = 15$ when $x = 5$, $y = 2$, and $z = 3$.

23. H varies jointly as x and y and inversely as $\sqrt{z}$; $H = 6$ when $x = 2$, $y = -3$, and $z = 9$.

24. If w is proportional to x and if $w = 5$ when $x = 7$, find w when $x = -6$.

25. If y is inversely proportional to x and if $y = 5$ when $x = 20$, find y when $x = 15$.

26. If H is proportional to x and inversely proportional to $\sqrt{y}$, and if $H = 3$ when $x = 2$ and $y = 4$, find H when $y = 9$ and $x = 5$.

27. The distance fallen by a body, starting from a position of rest in a vacuum near the earth's surface, is proportional to the square of the number of seconds occupied in falling. If a body falls 256 feet in 4 seconds, how far will it fall in 7 seconds?

28. The kinetic energy E, of a mass of m pounds moving with a velocity v, is proportional to mv^2. If $E = 2500$ foot-pounds when a body weighing 64 pounds is moving at a velocity of 50 feet per second, find the kinetic energy of a body weighing 30 pounds whose velocity is 2400 feet per minute.

29. If one body is sliding on another, the force of sliding friction is proportional to the normal pressure between the bodies (if this pressure is moderate). If the sliding friction between two cast-iron plates is 60 pounds when the normal pressure is 270 pounds, find the normal pressure when the sliding friction is 600 pounds.

30. The maximum safe load of a horizontal beam supported at its ends varies directly as its breadth and the square of its depth and inversely as the distance between the supports. If the maximum is 2400 pounds for a beam 4 inches wide and 10 inches deep, with supports 15 feet apart, find the maximum load for a beam of the same material which is 3 inches wide and 5 inches deep, with supports 25 feet apart.

31. How far apart may the supports be placed if a beam 5 inches wide and 8 inches deep, like those in Problem 30, supports 6000 pounds?

32. A beam like those in Problem 30 is 6 inches wide and the supports are 12 feet apart. How deep must the beam be to support 3500 pounds?

33. The approximate amount of steam per second which will flow through a hole varies jointly as the steam pressure and the area of a cross section of the hole. If 40 pounds of steam per second at a pressure of 200 pounds per square inch flows through a hole whose area is 14 square inches, (*a*) how much steam at a pressure of 250 pounds per square inch will flow through a hole whose area is 20 square inches; (*b*) what is the area of a hole which allows 30 pounds of steam to flow through it when the pressure is 300 pounds per square inch?

34. The electrical resistance of a wire varies as its length and inversely as the square of its diameter. If a wire 350 feet long and 3 millimeters in diameter has a resistance of 1.08 ohms, find the length of a wire of the same material whose resistance is .81 ohm and diameter is 2 millimeters.

35. If y is proportional to x and if $y = 16$ when $x = 4$, graph the relation between x and y. Make a statement about the change in the value of y, (*a*) if x varies from any given value to a value three times as large; (*b*) if x increases by 25% from a given value.

36. Repeat (*a*) and (*b*) of Problem 35 in case y is inversely proportional to x and $y = 16$ when $x = \frac{1}{4}$.

37. The approximate velocity of a stream of water, necessary to move a round object, is proportional to the product of the square roots of the object's diameter and its specific gravity. If a velocity of 11.34 feet per second is needed to move a stone whose diameter is 1 foot and specific gravity is 4, how large a stone with specific gravity 3 can be moved by a stream whose velocity is 22.68 feet per second?

38. Read Example 2 in Section 151. Find the ratio of the kinetic energy of a skater whose speed is 20 miles per hour to his energy when his speed is 15 miles per hour.

39. The horsepower that can be safely transmitted by a solid circular steel shaft varies jointly as the cube of its diameter and the number of revolutions it makes per minute. If a shaft 1.5″ in diameter rotating at 1520 revolutions per minute can transmit 135 horsepower, find the speed at which the shaft could transmit 162 horsepower.

40. The illumination received from a source of light varies inversely as the square of the distance from the source, and directly as its candle power. At what distance from a 50 candle power light would the illumination be one half that received at 30 feet from a 40 candle power light?

41. Newton's *Law of Gravitation* states that the force with which each of two masses of m pounds and M pounds attracts the other varies directly as the product of the masses and inversely as the square of the distance between the masses. Find the ratio of the force of attraction when two masses are 8000 miles apart to the force when they are 2000 miles apart.

42. As a first approximation, it is found that the wind pressure on a surface at right angles to the direction of the wind varies jointly as the area of the surface and the square of the wind velocity. What wind velocity would be necessary to cause the pressure on 40 square feet of surface to be double the pressure exerted on 10 square feet by a wind velocity of 30 miles per hour?

43. The current in an electric circuit varies directly as the electromotive force and inversely as the resistance. In a certain circuit, the electromotive force is A volts, the resistance is b ohms, and the current is c amperes. If the resistance is increased by 20%, what per cent of increase must occur in the voltage to increase the current by 30%?

Note 1. The statement *x is to y is to z as r is to s is to t*, or *x, y, and z are proportional to r, s, and t* is abbreviated by $x:y:z = r:s:t$, and means that there exists a number $k \neq 0$ such that $x = kr$, $y = ks$, and $z = kt$.

★*Find x, y, and z under the given conditions.*

44. $x:y:z = 4:-2:5$, and $x + 2y + z = 40$.

HINT. $x = 4k$; $y = -2k$; $z = 5k$. Substitute in the given equation and find the value of k.

45. $x:y:z = 5:-3:2$, and $x - y - z = 12$.

46. $x:y:z = 2:5:1$, and $x^2 + y^2 + z^2 = 120$.

47. $x:y:z = 3:-1:2$, and $x^2 + y^2 + z^2 = 56$.

48. Divide 2800 into four parts proportional to $5:3:4:2$.

49. Divide 1250 into four parts proportional to $3:5:11:6$.

CHAPTER 14

PROGRESSIONS

152. Arithmetic progressions

A *sequence* of things is a set of things arranged in a definite order. An *arithmetic progression* (abbreviated A.P.) is a sequence of numbers called *terms*, each of which, after the first, is derived from the preceding one by adding to it a fixed number called the **common difference.** The common difference can be found by subtracting any term from the one *following* it.

ILLUSTRATION 1. In the arithmetic progression $9, 6, 3, 0, -3, \cdots$, the common difference is -3. The 6th term would be -6.

153. The *n*th term in an arithmetic progression

Let a be the first term and d be the common difference. Then, the second term is $a + d$; the third term is $a + 2d$; the fourth term is $a + 3d$. In each of these terms, the coefficient of d is 1 *less than the number of the term.* Similarly, the tenth term is $a + 9d$. The nth term is the $(n - 1)$th after the first term, and is obtained after d has been added $(n - 1)$ times, in succession. Hence, if l represents the nth term,

$$l = a + (n - 1)d. \tag{1}$$

ILLUSTRATION 1. If $a = 3$ and $d = 4$, the 18th term is $3 + 17(4) = 71$.

154. Sum of an arithmetic progression

Let S be the sum of the first n terms of an A.P. The first term is a; the common difference is d; the last term is l; the next to the last term is $l - d$, etc. On writing the sum of the n terms, forward and backward, we obtain

$$S = a + (a + d) + (a + 2d) + \cdots + (l - 2d) + (l - d) + l; \quad (1)$$
$$S = l + (l - d) + (l - 2d) + \cdots + (a + 2d) + (a + d) + a. \quad (2)$$

On adding corresponding sides of (1) and (2) we obtain

$$2S = (a + l) + (a + l) + (a + l) + \cdots + (a + l) + (a + l) + (a + l),$$

where there are n terms $(a + l)$. Hence, $2S = n(a + l)$, or

$$S = \frac{n}{2}(a + l). \quad (3)$$

Example 1. Find the sum of the A.P. $8 + 5 + 2 + \cdots$ *to twelve terms.*

Solution. 1. First obtain l from $l = a + (n - 1)d$. We have

$a = 8$, $d = -3$, and $n = 12$: $\qquad l = 8 + 11(-3) = -25.$

2. From (3), $S = 6(8 - 25) = -102.$

If we rewrite (3) in the form

$$S = n\left(\frac{a + l}{2}\right), \quad (4)$$

we observe that the sum of an A.P. of n terms equals n times *the average of the first and last terms.*

On substituting $l = a + (n - 1)d$ in (3), we obtain

$$S = \frac{n}{2}[2a + (n - 1)d]. \quad (5)$$

The quantities a, d, l, n, and S are called the **elements** of the general arithmetic progression. When three of the elements are given, we may obtain the other two by use of $l = a + (n - 1)d$ and formulas 3 and 5.

Example 2. Find the remaining elements in an A.P. for which $a = 2$, $l = 402$, and $n = 26$.

Solution. 1. We wish to find d and S. From (3), $S = 13(404) = 5252.$

2. From $l = a + (n - 1)d$, $402 = 2 + 25d$; hence, $d = 16.$

If a sequence of three numbers a, b, and c forms an A.P., then $b - a = c - b$, because each side of this equation is equal to the common difference.

Example 3. Find the value of k if $(17, k, 29)$ form an A.P.

Solution. $\qquad k - 17 = 29 - k;\quad 2k = 46;\quad$ hence, $\quad k = 23.$

EXAMPLE 4. Find the sum of the A.P. $6 + 9 + 12 + \cdots + 171$.

SOLUTION. 1. We have given $a = 6$, $d = 3$, and $l = 171$.

2. To find n, use $l = a + (n - 1)d$:

$$171 = 6 + 3(n - 1); \quad 171 = 6 + 3n - 3; \quad n = 56.$$

3. To find S, use (3): $$S = \frac{56}{2}(6 + 171) = 4956.$$

EXAMPLE 5. Find the 39th term in an A.P. where the 4th term is -8 and the common difference is 5.

SOLUTION. 1. Think of a new A.P. where -8 is the 1st term; the former 39th term is the 36th term of the new progression.

2. Use $l = a + (n - 1)d$ with $a = -8$, $d = 5$, and $n = 36$:

$$\textit{desired } 39\text{th } \textit{term} = -8 + 35(5) = 167.$$

EXERCISE 82

Write the first six terms of an A.P. *from the given data.*

1. $a = 15;\ d = 3.$
2. $a = 17;\ d = -3.$
3. $a = -18;\ d = 2.$
4. $a = k;\ d = h.$

Which sequences do not form arithmetic progressions?

5. 3, 7, 11, 15.
6. 15, 17, 20, 22.
7. 23, 20, 17.
8. 35, 32, 30, 28.

Find the value of b for which the sequence forms an A.P.

9. 3, 8, b.
10. 25, 21, b.
11. 15, b, 13.
12. b, 17, 23.

Find the specified term of the A.P. *by use of a formula.*

13. Given terms: 4, 7, 10; find the 50th term.
14. Given terms: -5, -8, -11; find the 29th term.
15. Given terms: 4, 2, 0; find the 41st term.
16. Given terms: 3, $3\frac{1}{4}$, $3\frac{1}{2}$; find the 83d term.
17. Given terms: 2.4, 2.6, 2.8; find the 39th term.
18. Given terms: 3, 2.95, 2.9; find the 201st term.

Find the last term and the sum of the A.P. *by use of formulas.*

19. 8, 13, 18, $\cdots$ to 15 terms.
20. 3, 5, 7, $\cdots$ to 41 terms.
21. 9, 6, 3, $\cdots$ to 28 terms.
22. 13, 8, 3, $\cdots$ to 17 terms.
23. 2.06, 2.02, 1.98, $\cdots$ to 33 terms.
24. 5, $4\frac{1}{2}$, 4, $\cdots$ to 81 terms.

In each problem, certain of the elements a, d, l, n, and S are given. Find the missing elements.

25. $a = 10, l = 410, n = 26.$

26. $a = 4, l = 72, n = 18.$

27. $a = 17, l = 381, d = 4.$

28. $l = 53, d = 4, n = 19.$

29. $l = 87, d = -3, n = 18.$

30. $a = 27, l = 11, d = -\frac{1}{4}.$

31. $a = 50, l = 0, d = -\frac{5}{2}.$

32. $S = -2496, n = 52, a = 3.$

33. $S = 2337, n = 38, d = \frac{7}{2}.$

34. $n = 26, S = 5278, d = 16.$

Find the value of k for which the sequence of three terms forms an A.P.

35. $(3 - 2k);\ (2 - k);\ (4 + 3k).$

36. $(2 + k);\ (2 + 4k);\ (5k - 1).$

37. Find the 45th term in an A.P. where the 3d term is 7 and the common difference is $\frac{1}{3}$.

38. Find the 59th term in an A.P. where the 4th term is 9 and the common difference is $-.4$.

39. In the A.P. .97, 1.00, 1.03, $\cdots$, which term is 5.02?

40. In the A.P. 16, 13.5, 11, $\cdots$, which term is -129?

41. Find the common difference of an A.P. whose 6th term is 9 and 37th term is 54.

155. Arithmetic means

The first term, a, and the last term, l, in an arithmetic progression are called the *extremes* of the progression. The other terms are called *arithmetic means* between a and l. *To insert k arithmetic means* between two numbers, a and l, means to find a sequence of k numbers which, when placed between a and l, give rise to an A.P. with a and l as its extremes.

Example 1. Insert five arithmetic means between 13 and -11.

Solution. 1. After the means are inserted, they will complete an A.P. of seven terms, with $a = 13$ and $l = -11$. We shall find d for the progression and then form the terms.

2. From $l = a + (n - 1)d$,

$$-11 = 13 + 6d; \qquad d = -4.$$

3. Hence, the missing terms are $(13 - 4)$, or 9; $(9 - 4)$, or 5; etc. The A.P. is $(13, 9, 5, 1, -3, -7, -11)$. Therefore the arithmetic means are $(9, 5, 1, -3, -7)$.

When a *single* arithmetic mean is inserted between two numbers, it is called the *arithmetic mean* of the numbers. Thus, if (b, A, c) form an A.P., then A is called the arithmetic mean of b and c. Then, $A - b = c - A$ or $2A = c + b$. Hence,

$$A = \frac{b + c}{2}, \qquad (1)$$

or the arithmetic mean of two numbers is *one half of their sum*. Thus, the arithmetic mean of b and c is the number which is frequently called the *average* of b and c.

ILLUSTRATION 1. The arithmetic mean of 7 and 15 is $\frac{1}{2}(7 + 15) = 11$.

Note 1. The *average* of k numbers is defined as *their sum divided by k*. As a generalization of equation 1, the average of k numbers is frequently called the *arithmetic mean* of the numbers. Unless we are dealing with just *two* numbers, so that $k = 2$, the arithmetic mean of k numbers has *no connection with the notion of arithmetic means* as they occur in arithmetic progressions.

156. Applications of arithmetic progressions

In a problem dealing with an A.P., write down the first few terms of the progression and describe them in the language of the problem. Then, decide which elements are known and which you wish to find.

EXAMPLE 1. A man invests \$1000 at the end of each year for 30 years at 6% simple interest. Find the accumulated value of his investments at the end of 30 years, if no interest is withdrawn until then.

SOLUTION. 1. The first \$1000 invested will draw interest at 6% for 29 years, or a total of \$1740 interest; the resulting amount at the end of 30 years is \$2740.

2. The second \$1000 invested will draw interest for 28 years; the resulting interest is \$1680 and the amount at the end of 30 years is \$2680.

3. Etc.; the \$1000 invested at the end of 29 years will draw interest for just one year; the resulting amount is \$1060. The last \$1000 is invested at the end of 30 years and receives no interest.

4. The total amount at the end of 30 years is

$$2740 + 2680 + 2620 + \cdots + 1060 + 1000.$$

We wish S for an A.P. in which $a = 2740$, $n = 30$, and $d = -60$.

5. From $S = \frac{n}{2}(a + l)$, $\qquad S = \frac{30}{2}(2740 + 1000) = \$56{,}100.$

EXAMPLE 2. A contractor has agreed to pay a penalty if he uses more than a specified length of time to finish a certain job. The penalties for excess time are \$25 for the 1st day and, thereafter, \$5 more for each day than for the preceding day. If he pays a total penalty of \$4050, how many excess days did he need to finish the work?

SOLUTION. 1. The penalties are \$25, \$30, \$35, $\cdots$, which form an A.P. where $a = 25$, $d = 5$, and $S = 4050$. We wish to find the number of terms, n.

2. From $\quad S = \frac{1}{2}n[2a + (n - 1)d]$,

$$4050 = \frac{n}{2}[50 + 5(n - 1)]. \tag{1}$$

3. To solve (1), multiply both sides by 2:

$$8100 = 50n + 5n^2 - 5n;$$

$$n^2 + 9n - 1620 = 0. \tag{2}$$

On solving (2) by factoring, or the quadratic formula, we find $n = 36$ and $n = -45$. The negative root has no application in the problem. Hence, there are 36 excess days.

CHECK. The student should compute the sum of

$$25 + 30 + 35 + \cdots \textit{ to 36 terms.}$$

EXERCISE 83

1. Insert four arithmetic means between 2 and 17.
2. Insert five arithmetic means between -2 and 40.
3. Insert five arithmetic means between 7 and 17.
4. Insert four arithmetic means between 19 and 12.
5. Insert six arithmetic means between 15 and -16.5.
6. Insert seven arithmetic means between $\frac{5}{2}$ and 7.
7. Insert five arithmetic means between $-\frac{3}{2}$ and 6.

Find the arithmetic mean of the numbers.

8. 6; 38. **9.** 15; 37. **10.** -13; 27. **11.** -15; -23. **12.** x; y.

13. Find the sum of all even integers from 10 to 380 inclusive.
14. Find the sum of all odd integers from 15 to 361 inclusive.
15. Find the sum of the first 38 positive integral multiples of 3.
16. Find the sum of all positive integral multiples of 5 which are less than 498.

17. There are 16 rows of billiard balls in a symmetrical triangular arrangement on a table, with 46 balls in the first row and 3 less balls in each other row than in the one preceding it. How many balls are on the table?

18. Find the sum of all positive and negative integral multiples of 6 between -55 and 357.

19. The horizontal base of a right triangle is 15 feet long and the side perpendicular to this base is 45 feet long. At intervals of 1 foot on the base, a perpendicular is drawn to the base and reaches to the hypotenuse. Find the sum of the lengths of all perpendiculars, including the vertical leg of the triangle.

20. A man invests $1000 at the end of each year for 12 years at 6% simple interest. What is the accumulated value of his investments at the end of 12 years?

Find the total sum of money paid by the debtor in discharging his debt.

21. Debtor borrows $10,000. *Agrees to pay:* at the end of each year for 10 years, $1000 principal and simple interest at 3% on all principal outstanding during the year.

22. Debtor borrows $20,000. *Agrees to pay:* at the end of each year for 20 years, $1000 principal and simple interest at 5% on all principal outstanding during the year.

23. A man invests $1000 at the beginning of each year for 20 years at 5% simple interest. Find the accumulated value of his investments at the end of 20 years.

24. The 4th term of an A.P. is 215 and the 44th term is 55. Find the sum of the first 20 terms.

25. If $y = 5x + 8$, find the sum of the values of y corresponding to the successive integral values $x = 1, 2, 3, \cdots, 30$.

26. The bottom rung of a ladder is 28 inches long and each other rung is one half inch shorter than the rung below it. If the ladder has 18 rungs, how many feet of wood were used in making the rungs?

157. Geometric progressions

A *geometric progression* (abbreviated G.P.) is a sequence of numbers called *terms,* each of which, after the first, is obtained by *multiplying* the preceding term by a fixed number called the **common ratio.** The common ratio equals the *ratio of any term, after the first, to the one preceding it.*

ILLUSTRATION 1. In the G.P. 16, -8, $+4$, -2, $\cdots$, the common ratio is $-\frac{1}{2}$; the 5th term would be $(-\frac{1}{2})(-2) = +1$.

To determine whether or not a sequence of numbers forms a geometric progression, we *divide* each number by the one which precedes it. All of these ratios are equal if the terms form a G.P.

ILLUSTRATION 2. If 3, 8, and x form a G.P., then $\frac{8}{3} = \frac{x}{8}$ or $x = \frac{64}{3}$.

If the terms of a G.P. are *reversed*, the terms will form a G.P. whose common ratio is the *reciprocal* of the ratio for the given G.P.

ILLUSTRATION 3. In the G.P. (4, 8, 16, 32), the common ratio is 2. When the terms are reversed, we have (32, 16, 8, 4), where the ratio is $\frac{1}{2}$.

ILLUSTRATION 4. The G.P. (a, ar, ar^2, ar^3) has the common ratio r whereas the G.P. (ar^3, ar^2, ar, a) has the common ratio ar^2/ar^3 or $1/r$.

158. The nth term of a geometric progression

Let a be the first term and r be the common ratio. Then, the second term is ar; the third term is ar^2. In each of these terms the exponent of r is 1 *less than the number of the term.* Similarly, the eighth term is ar^7. The nth term is the $(n - 1)$th after the 1st and hence is found by multiplying a by $(n - 1)$ factors r, or by r^{n-1}. Hence, if l represents the nth term,

$$l = ar^{n-1}. \tag{1}$$

ILLUSTRATION 1. If $a = 3$ and $r = 2$, the 7th term is $3(2^6) = 192$.

159. Sum of a geometric progression

Let S be the sum of the first n terms of a G.P. The terms are $(a, ar, ar^2, \cdots, ar^{n-2}, ar^{n-1})$, where ar^{n-2} is the $(n - 1)$th term.

Hence, $$S = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}, \tag{1}$$

and $$Sr = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n; \tag{2}$$

in (2) we multiplied both sides of (1) by r. On subtracting each side of (2) from the corresponding side of (1), we obtain

$$S - Sr = a - ar^n, \tag{3}$$

because each term, except ar^n, on the right in (2) cancels a corresponding term in (1). From (3), $S(1 - r) = a - ar^n$, or

$$S = \frac{a - ar^n}{1 - r}. \tag{4}$$

Since $l = ar^{n-1}$, then $rl = ar^n$. Hence, from (4),

$$S = \frac{a - rl}{1 - r}. \tag{5}$$

In using (4), it is sometimes convenient to rewrite it as

$$S = a\frac{1 - r^n}{1 - r}. \tag{6}$$

EXAMPLE 1. Find the sum of the G.P. 2, 6, 18, $\cdots$ to six terms.

SOLUTION. $n = 6$; $a = 2$; $r = 3$. From (4),

$$S = \frac{2 - 2\cdot 3^6}{1 - 3} = \frac{2 - 1458}{-2} = 728.$$

Formula 5 is convenient when l is explicitly given.

EXAMPLE 2. Find the sum of the geometric progression

$$(1.05)^2 + (1.05)^5 + (1.05)^8 + \cdots + (1.05)^{35}.$$

SOLUTION. $a = (1.05)^2$; $r = (1.05)^3$; $l = (1.05)^{35}$. From formula 5,

$$S = \frac{(1.05)^2 - (1.05)^3(1.05)^{35}}{1 - (1.05)^3} = \frac{(1.05)^2 - (1.05)^{38}}{1 - (1.05)^3}.$$

Note 1. When a sufficient number of the *elements* (a, r, n, l, S) are given, we find the others by use of $l = ar^{n-1}$, (4), and (5).

EXAMPLE 3. If $S = 750$, $r = 2$, and $l = 400$, find n and a.

SOLUTION. 1. From $S = \dfrac{a - rl}{1 - r}$,

$$750 = \frac{a - 800}{1 - 2}; \quad \text{hence,} \quad a = 50.$$

2. From $l = ar^{n-1}$, $\qquad 400 = 50(2^{n-1})$; $\quad 2^{n-1} = \dfrac{400}{50} = 8$;

$$2^{n-1} = 2^3; \quad \text{hence,} \quad n - 1 = 3, \quad \text{or} \quad n = 4.$$

If three numbers (a, b, c) form a G.P., then $\dfrac{b}{a} = \dfrac{c}{b}$.

ILLUSTRATION 1. If $(a, 10, 50)$ form a G.P. then $\dfrac{10}{a} = \dfrac{50}{10}$ or $a = 2$.

160. Geometric means

The first term, a, and the last term, l, in a G.P. are called the *extremes* of the progression. The other terms are called *geometric means* between a and l. *To insert k geometric means* between two numbers, a and l, means to find a sequence of k numbers which, when placed between a and l, give rise to a G.P. with a and l as its extremes. In asking for geometric means, we shall desire only *real valued* means.

EXAMPLE 1. Insert two geometric means between 6 and $\frac{16}{9}$.

SOLUTION. After the means are inserted, they will complete a G.P. of four terms with $a = 6$ and $l = \frac{16}{9}$. We shall find the common ratio of the progression, and then its two middle terms. From $l = ar^{n-1}$, with $n = 4$, we obtain

$$\frac{16}{9} = 6r^3; \quad r^3 = \frac{8}{27}; \quad r = \sqrt[3]{\frac{8}{27}} = \frac{2}{3}.$$

The G.P. is $(6, 4, \frac{8}{3}, \frac{16}{9})$. The geometric means are 4 and $\frac{8}{3}$.

EXERCISE 84

Write the first four terms of a G.P. *for the given data.*

1. $a = 5$; $r = 3$.
2. $a = 16$; $r = \frac{1}{2}$.
3. $a = 4$; $r = -2$.
4. $a = 27$; $r = -\frac{1}{3}$.

In case the numbers form a G.P., *state its common ratio and write two more terms of the* G.P.

5. $4, -12, 36, 108$.
6. $10, 5, \frac{5}{2}, \frac{5}{4}$.
7. $8, \frac{8}{3}, \frac{8}{9}, \frac{8}{27}$.
8. $4, 2, 1, 0$.
9. a, ax, ax^2, ax^3.
10. $(1.02)^4, (1.02)^6, (1.02)^8$.
11. $(1.01)^{-5}, (1.01)^{-3}, (1.01)^{-1}$.

Find the value of x so that the three numbers form a G.P.

12. $5, 20, x$.
13. $x, 12, 36$.
14. $4, x, 16$.
15. $x, -4, 10$.
16. If the G.P. (81, 27, 9, 3) is reversed, what fact do you observe?

By use of $l = ar^{n-1}$, find the specified term of the given G.P. *without finding the intermediate terms.*

17. 3, 9, 27; find the 6th term.
18. $4, -12, 36$; find the 9th term.
19. 12, 6, 3; find the 8th term.
20. $6, -3, \frac{3}{2}$; find the 9th term.

Find the last term and the sum of the G.P.

21. 4, 12, 36, $\cdots$ to 7 terms.

22. 12, 6, 3, $\cdots$ to 6 terms.

23. 5, $-$ 15, 45, $\cdots$ to 6 terms.

24. 25, 2.5, .25, $\cdots$ to 7 terms.

25. 3, $-$ 6, 12, $\cdots$ to 7 terms.

26. $\frac{1}{16}$, $-\frac{1}{4}$, 1, $\cdots$ to 6 terms.

27. 3, $6b$, $12b^2$, $\cdots$ to 8 terms.

28. 4, $8x^2$, $16x^4$, $\cdots$ to 7 terms.

Employ formula 5 *on page* 229 *to find the sum of the* G.P.

29. $4 + 2 + \cdots + \frac{1}{128}$.

30. $5 + 15 + \cdots + 3645$.

Find the missing elements of the G.P.

31. $a = 5$; $r = 2$; $l = 640$.

32. $a = 2$; $r = 3$; $l = 486$.

33. $r = 10$; $a = .001$; $l = 100$.

34. $S = 2186$; $l = 1458$; $a = 2$.

35. $S = 275$; $r = -2$; $l = 400$.

36. $S = \frac{910}{9}$; $a = -\frac{5}{9}$; $l = 135$.

37. $a = 256$; $r = \frac{1}{2}$; $l = \frac{1}{4}$.

38. $a = 1458$; $r = \frac{1}{3}$; $l = \frac{2}{3}$.

Find the specified term without finding the first term of the G.P.

39. The 10th term, if the 6th term is 5 and common ratio is 2.

40. The 12th term, if the 8th term is 25 and common ratio is .1.

41. The 4th term, if the 8th term is 40 and common ratio is 2.

42. The 5th term, if the 9th term is 80 and common ratio is $\frac{1}{2}$.

Insert the specified number of geometric means.

43. Five, between 2 and 128.

44. Five, between 128 and 2.

45. Three, between 4 and 324.

46. Four, between $\frac{1}{3}$ and 81.

47. Six, between .1 and 1,000,000.

48. Three, between 16 and .0001.

If x and y are of the same sign, and if a single geometric mean G of the same sign is inserted between x and y, then G is called **the geometric mean** *of x and y;* (x, G, y) *form a* G.P. *Find the geometric mean of the numbers*

49. $\frac{1}{4}$; 16.

50. $\frac{1}{9}$; 36.

51. 4; 25.

52. $-$ 9; $-$

53. Find the geometric mean of x and y. State the result in words

Find an expression for the sum and simplify by use of the laws of ex but do not compute. Use formula 5 *on page* 229 *when convenient.*

54. $1 + (1.03) + (1.03)^2 + \cdots + (1.03)^{40}$.

55. $1 + (1.05) + (1.05)^2 + \cdots + (1.05)^{56}$.

56. $(1.02)^2 + (1.02)^3 + (1.02)^4 + \cdots + (1.02)^{37}$.

57. $(1.06)^4 + (1.06)^5 + (1.06)^6 + \cdots + (1.06)^{29}$.

58. $1 + (1.02)^3 + (1.02)^6 + \cdots$ to 21 terms.

59. $(1.02)^{-15} + (1.02)^{-14} + (1.02)^{-13} + \cdots + (1.02)^{-1}$.

60. $(1.03)^{-16} + (1.03)^{-14} + (1.03)^{-12} + \cdots + (1.03)^{-4}$.

61. $1 + (1.02)^{\frac{1}{2}} + (1.02) + (1.02)^{\frac{3}{2}} + \cdots + (1.02)^{\frac{12}{2}}$.

62. If the 7th term of a G.P. is 5 and the 11th term is $\frac{5}{16}$, find the intermediate terms.

63. For what values of k do the three quantities $(k + 3)$, $(6k + 3)$, and $(20k + 5)$ form a G.P.?

64. Find the sum of a G.P. of 7 terms whose 3d term is $\frac{5}{4}$ and 6th is $\frac{5}{32}$.

65. How many ancestors have you had in the twelve preceding generations if no ancestor appears in more than one line of descent?

66. An investment paid a man, in each year after the first, twice as much as in the preceding year. If his investment paid him \$13,500 in the first four years, how much did it pay the investor in the first and the fourth years?

67. In a lottery, it is agreed that the first ticket drawn will pay its owner \$.10 and each succeeding ticket twice as much as the preceding one. Find the total amount paid on the first 10 tickets drawn.

68. Find the sum of the first 19 positive integral powers of 1.03, given that $(1.03)^{10} = 1.344$.

161. Applications of geom ric progressions

When a problem is met where a sequence of terms is suspected of forming A.P., generally it is an advantage to compute the *explicit values* of the first few terms in *simplest form* in order to verify the istence of a *common difference* between the terms. On the other d, if a sequence of terms is suspected of forming a G.P., it is best ite the first few terms, *without actually computing them*, in a hich will exhibit clearly any *constant factor* which appears to e powers.

1. A rubber ball is dropped from a height of 100 feet. On l, the ball rises one half of the height from which it last fell. has the ball traveled up to the instant it hits the ground for

SOLUTION. 1. We list the first few distances traveled by the bouncing ball:

$$\begin{aligned}
&\text{1st } \textit{fall} = 100 \text{ ft.} \\
&\left.\begin{aligned}\text{1st } \textit{rise} &= \tfrac{1}{2}(100) \text{ ft.}\\ \text{2d } \textit{fall} &= \tfrac{1}{2}(100) \text{ ft.}\end{aligned}\right\} \quad \textit{sum} = 100 \text{ ft.}\\
&\left.\begin{aligned}\text{2d } \textit{rise} &= \tfrac{1}{2}(\tfrac{1}{2})(100) \text{ ft.} = \tfrac{1}{4}(100) \text{ ft.}\\ \text{3d } \textit{fall} &= \tfrac{1}{4}(100) \text{ ft.}\end{aligned}\right\} \quad \textit{sum} = \tfrac{1}{2}(100) \text{ ft.}\\
&\left.\begin{aligned}\text{3d } \textit{rise} &= \tfrac{1}{2}(\tfrac{1}{4})(100) \text{ ft.}\\ \text{4th } \textit{fall} &= \tfrac{1}{2}(\tfrac{1}{4})(100) \text{ ft.}\end{aligned}\right\} \quad \textit{sum} = \tfrac{1}{4}(100) \text{ ft.}\\
&\qquad \textit{etc.} \qquad\qquad\qquad\qquad \textit{etc.}
\end{aligned}$$

2. The 1st fall brings in an unsymmetrical term. On neglecting it temporarily, the total distance, in feet, traveled otherwise up to the time of the 12th fall is the sum of

$$100,\ \tfrac{1}{2}(100),\ \tfrac{1}{4}(100),\ \cdots \textit{ to eleven terms.} \tag{1}$$

In (1) we have a G.P. with $a = 100$, $r = \frac{1}{2}$, and $n = 11$. The sum S is obtained from

$$S = a\frac{1 - r^n}{1 - r} = 100\frac{1 - (\frac{1}{2})^{11}}{1 - \frac{1}{2}} = 100\frac{1 - \frac{1}{2048}}{\frac{1}{2}} = \frac{100(2047)}{1024};$$

$$S = \frac{25(2047)}{256} = 199\tfrac{231}{256}.$$

3. On adding the 1st fall to S, we find that the total distance traveled by the ball is $299\frac{231}{256}$ feet.

MISCELLANEOUS EXERCISE 85

Solve by methods involving progressions.

1. If \$500 is to be divided between 10 men so that the first one receives \$5 and each succeeding man obtains a fixed amount more than the preceding man, how much will the 10th man receive?

2. At a bazaar, tickets are marked with the consecutive even integers 2, 4, 6, $\cdots$ and are drawn at random by those entering. If each person pays as many cents as the number on his ticket, how much money is received if 1000 tickets are sold?

3. The path of each swing, after the first, of a pendulum bob is .9 as long as the preceding swing. If the first swing is 40 inches long, how far does the pendulum travel on the first 8 swings?

4. A man piles 152 logs in layers so that the top layer contains 2 logs and each lower layer has one more log than the layer above. How many logs will be in the lowest layer?

5. In a professional golf tournament, the total prize money of $5187 is divided among the six players with lowest scores, so that each man above the lowest receives $\frac{2}{5}$ as much as the man below him. How much does the man with the lowest score receive?

6. A body, dropped from a position of rest in a vacuum near the earth's surface, will fall approximately 32 feet farther in each second, after the first, than in the preceding second. If a body falls 10,000 feet in 25 seconds, how far does it fall in the first second?

7. Find an expression for the sum of the first n positive integers.

8. Find an expression for the sum of the first n positive even integers.

9. At the beginning of each year, a man invests $300 at simple interest at the rate 7%. At the end of 15 years, what is the total value of his investments if none of them have been disturbed, and if all required interest is paid on that date?

10. The radiator of a motor truck contains 10 gallons of water. We draw off 1 gallon and replace it with alcohol; then, we draw off 1 gallon of the mixture and replace it by alcohol; etc., until 9 drawings and replacements have been made. How much alcohol is in the final mixture?

11. Find the sum of the first 40 positive integral powers of x.

12. In creating a vacuum in a container, a pump draws out $\frac{1}{4}$ of the remaining air at each stroke. What part of the original air has been removed by the end of the 7th stroke?

13. A pendulum bob moves over a path 15 inches long on its first swing. In each succeeding swing the bob travels four fifths of the distance of the preceding swing. How far does the bob travel during the first six swings?

14. In a potato race, twenty potatoes are placed at intervals of 5 feet in a line from the starting point, with the nearest potato 25 feet away. A runner is required to bring the potatoes back to the starting place one at a time. How far would he run in bringing in all the potatoes?

15. A speculator will make $1200 during the first month and, thereafter, in each month, $100 less than in the preceding month. If his original capital is $2700, when will he become bankrupt?

16. Two men start in a distance run. One man proceeds at a uniform speed of 300 yards per minute. The second man travels 435 yards in the first minute, but, thereafter, in each minute he goes 30 yards less than in the preceding minute. When will the first man overtake the second?

17. Prove that the squares of the terms of a G.P. also form a G.P. Then state a more general theorem of this nature.

18. Prove that the reciprocals of the terms of a G.P. also form a G.P.

19. A rubber ball is dropped from a height of 300 feet. On each rebound, the ball rises one third of the height from which it last fell. What distance has the ball traveled up to the instant the ball hits the ground for the 7th time?

20. In a certain positive integral number of three digits, the digits form an A.P. and their sum is 15. If the digits are reversed, the new number is 594 less than the original number. Find the original number.

Note 1. If P is the value of a certain quantity *now*, and if its value increases at the rate i (expressed as a decimal) per year, then the new value at the end of one year is $(P + Pi)$, or $P(1 + i)$. That is, *the value at the end of any year is* $(1 + i)$ *times the value at the end of the last year.* The values at the ends of the years form a G.P. whose common ratio is $(1 + i)$. If A represents the value at the end of n years, then

$$A = P(1 + i)^n.$$

This formula is referred to as the **compound interest law** because, if a principal P is invested now at the rate i, compounded annually, the amount A at the end of n years will be $P(1 + i)^n$. Applications involving compound interest will be treated later. In all of the following problems, it will be assumed that any *rate* is *constant.*

21. If 300 units of a commodity are consumed in a first year, and if the annual rate of increase of consumption is 6%, (*a*) give an expression for the amount consumed in the 7th year; (*b*) find the total consumption in the first 12 years, given that $(1.06)^{12} = 2.012$.

22. A corporation will sell $1,000,000 worth of its products this year and the sales will increase at the rate of 5% per year. Find the total sales during the first 25 years, given that $(1.05)^{25} = 3.38635494$.

23. The population of a city increased from 131,220 to 200,000 in 4 years. Find the rate of increase per year.

24. A piece of property was purchased 4 years ago for $4860 and its value now is $15,360. Find the annual rate at which the value increased.

25. The value of a certain quantity *decreases* at the rate w (expressed as a decimal) per year. If H is the value now, and K is the value at the end of n years, prove that $K = H(1 - w)^n$. (This formula is the basis for computing depreciation charges in business under the so-called *constant-percentage* method.)

26. A motor truck was purchased for $2500, and its value 4 years later is $1024. Find the rate per year at which the value has depreciated.

27. A hotel, purchased 3 years ago for $512,000, is sold for $343,000. Find the rate per year at which its value has depreciated.

★162. Harmonic progressions

A sequence of numbers is said to form a *harmonic* * *progression* if their reciprocals form an *arithmetic progression.*

ILLUSTRATION 1. The sequence $(1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9})$ is a harmonic progression because the reciprocals (1, 3, 5, 7, 9) form an A.P.

To insert k harmonic means between two numbers, we first insert *k arithmetic means* between the *reciprocals of the numbers.* The reciprocals of the arithmetic means are the harmonic means.

EXAMPLE 1. Insert five harmonic means between 4 and 16.

SOLUTION. 1. First, we insert 5 arithmetic means between $\frac{1}{4}$ and $\frac{1}{16}$.

2. From $l = a + (n - 1)d$, with $a = \frac{1}{4}$, $l = \frac{1}{16}$, and $n = 7$, we find

$$\tfrac{1}{16} = \tfrac{1}{4} + 6d; \quad d = -\tfrac{1}{32}.$$

3. Hence, the A.P. is $(\frac{1}{4}, \frac{7}{32}, \frac{6}{32}, \frac{5}{32}, \frac{4}{32}, \frac{3}{32}, \frac{1}{16})$.

4. The corresponding harmonic progression is $(4, \frac{32}{7}, \frac{16}{3}, \frac{32}{5}, 8, \frac{32}{3}, 16)$.

Hence, the harmonic means are $(\frac{32}{7}, \frac{16}{3}, \frac{32}{5}, 8, \frac{32}{3})$.

★EXERCISE 86

Insert the specified number of harmonic means.

1. Four, between $\frac{1}{2}$ and $\frac{1}{12}$.
2. Five, between $\frac{1}{4}$ and $\frac{1}{28}$.
3. Four, between $\frac{5}{14}$ and $\frac{5}{4}$.
4. Four, between 4 and 24.
5. Five, between $\frac{5}{3}$ and $\frac{1}{3}$.
6. Four, between $-\frac{1}{3}$ and 3.

If (c, H, d) form a harmonic progression, then H is called **the harmonic mean** *of c and d. Find the harmonic mean of the numbers.*

7. 4; $\frac{4}{3}$. **8.** 9; 6. **9.** 4; -8. **10.** 12; 36. **11.** x and y.

★163. Geometric progressions with infinitely many terms

Let S_n represent the sum of the progression $a, ar, ar^2, \cdots, ar^{n-1}$. Then, by (4), page 229,

$$a + ar + ar^2 + \cdots + ar^{n-1} = S_n = \frac{a}{1-r} - \frac{ar^n}{1-r}. \tag{1}$$

* Suppose that a set of strings of the same diameter and substance are stretched to uniform tension. If the lengths of the strings form a harmonic progression, a harmonious sound results if two or more strings are caused to vibrate at one time. This fact accounts for the name *harmonic progression.*

ILLUSTRATION 1. Consider the endless geometric progression

$$1, \frac{1}{2}, \frac{1}{4}, \cdots, \frac{1}{2^{n-1}}, \cdots \textit{ to infinitely many terms.} \tag{2}$$

In (2), $\quad r = \frac{1}{2};\quad$ the nth term is $\frac{1}{2^{n-1}};\quad 1 - r = \frac{1}{2};\quad ar^n = \frac{1}{2^n}.$

By (1), $$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{n-1}} = S_n = 2 - \frac{1}{2^{n-1}}. \tag{3}$$

If n grows larger, without limit, the term $\frac{1}{2^{n-1}}$ grows smaller, and is as near to zero as we please, if n is sufficiently large. Thus, if $n = 65$,

$$\frac{1}{2^{n-1}} = \frac{1}{2^{64}} = \frac{1}{18{,}446{,}744{,}073{,}709{,}551{,}616},$$

which is practically zero. Hence, in (3), S_n will be as near to $(2 - 0)$ as we please for all values of n which are sufficiently large. To summarize this statement we say that *as n becomes infinite S_n approaches the* **limit** 2, and we call 2 the *sum* of the progression $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$ *to infinitely many terms.* We sometimes use "$n \to \infty$" to abbreviate "*n becomes infinite.*" Then, our conclusion can be briefly written $\underset{n\to\infty}{\textit{limit}}\, S_n = 2$.

Now, consider ($a, ar, ar^2, \cdots$ *to infinitely many terms*), under the condition that r is a number between -1 and $+1$. Then, as $n \to \infty$, the absolute value of the numerator ar^n in (1) grows smaller, and is as near to zero as we please for all values of n sufficiently large. Hence, from (1) we see that, as n grows large without limit, the value of S_n approaches

$$\left(\frac{a}{1-r} - \frac{0}{1-r}\right), \quad \text{or} \quad \frac{a}{1-r}.$$

That is, $$\underset{n\to\infty}{\textit{limit}}\, S_n = \frac{a}{1-r}. \tag{4}$$

This limit of the sum of n terms, as n becomes infinite, is called the **sum** of the geometric progression with infinitely many terms. If S represents this sum, then

$$S = \frac{a}{1-r}. \tag{5}$$

Thus, if $|r| < 1$,

$$(a + ar + ar^2 + \cdots \textit{ to infinitely many terms}) = \frac{a}{1-r}. \tag{6}$$

Note 1. Recognize that S in (5) is *not* a sum in the ordinary sense of the word, but is *the limit of the sum of n terms as n grows large without bound.*

ILLUSTRATION 2. By use of (6), with $a = 5$ and $r = \frac{1}{2}$,

$$\left(5 + \frac{5}{2} + \frac{5}{4} + \cdots \textit{ to infinitely many terms}\right) = \frac{5}{1 - \frac{1}{2}} = 10.$$

Practically, this means that, if we should add a relatively large number of terms, we would obtain approximately 10, and that by adding enough terms we can obtain as close to 10 as we may desire. Thus, $S_{11} = 9\frac{1019}{1024}$.

The indicated sum of a sequence of numbers is frequently called a **series.** An expression of the form

$$u_1 + u_2 + u_3 + \cdots \textit{ to infinitely many terms} \tag{7}$$

is called an **infinite series.** Accordingly, the expression on the left in (6) is referred to as the *infinite geometric series.*

EXAMPLE 1. Find a rational number equal to the endless repeating decimal $.5818181 \cdots$.

SOLUTION. Because of the meaning of the decimal notation,

$$.5818181 \cdots = .5 + .081 + .00081 + \cdots \textit{ to infinitely many terms,}$$

where we notice that $(.081 + .00081 + \cdots)$ is an infinite geometric series with $a = .081$, and $r = .01$. By (6), the sum of this series is

$$\frac{.081}{1 - .01} = \frac{.081}{.99} = \frac{9}{110}.$$

Hence, $$.5818181 \cdots = .5 + \tfrac{9}{110} = \tfrac{5}{10} + \tfrac{9}{110} = \tfrac{32}{55}.$$

Comment. By use of the method of Example 1, we can show that *any endless repeating decimal is a rational number.*

In any infinite series such as (7), let S_n represent the sum of the first n terms. Then we say that the series *has a sum S,* and call the series a **convergent infinite series** which *converges* to S in case *the limit of S_n is S as n becomes infinite.* If S_n has no limit as n becomes infinite, we say that the infinite series is **divergent,** or *diverges.*

Note 2. In this section we have proved that the infinite geometric series in parentheses in (6) has a sum, or *converges,* when $|r| < 1$. When $|r| \geqq 1$, the series is divergent, or *does not have a sum,* because in this case S_n in (1) does not approach a limit as $n \to \infty$. Thus, for the G.P. $(1, 2, 4, \cdots)$ where $r = 2$, we find that S_n increases beyond all bounds as $n \to \infty$.

★EXERCISE 87

Find the sum of each of the following infinite geometric series by use of the established formula.

1. $7 + \frac{7}{2} + \frac{7}{4} + \cdots$.

2. $12 + 3 + \frac{3}{4} + \cdots$.

3. $15 + 5 + \frac{5}{3} + \cdots$.

4. $10 - 5 + \frac{5}{2} + \cdots$.

5. $1 - \frac{1}{3} + \frac{1}{9} - \cdots$.

6. $1 - \frac{1}{2} + \frac{1}{4} - \cdots$.

7. $1 - .01 + .0001 - \cdots$.

8. $.8 + .08 + .008 + \cdots$.

Find a rational number equal to the given endless repeating decimal, where repeating parts are written three times.

9. $.333\cdots$.

10. $.444\cdots$.

11. $.666\cdots$.

12. $.0999\cdots$.

13. $.8333\cdots$.

14. $.1666\cdots$.

15. $.212121\cdots$.

16. $.050505\cdots$.

17. $.030303\cdots$.

18. $.838383\cdots$.

19. $.454545\cdots$.

20. $4.222\cdots$.

21. $.2111\cdots$.

22. $.345345345\cdots$.

23. $.210210210\cdots$.

24. $252.525\cdots$.

25. $16.7167167\cdots$.

26. $25.05050\cdots$.

27. $.153846153846153846\cdots$.

28. $.076923076923076923\cdots$.

29. A pendulum is being brought to rest by air resistance. The path of each swing, after the first, of the pendulum bob is .98 as long as the path of the previous swing (from one side to the other). If the path of the first swing is 30 inches long, how far does the bob travel in coming to a position of rest?

30. A rubber ball is dropped from a height of 100 inches. On each rebound the ball rises to $\frac{5}{6}$ of the height from which it last fell. Find the distance traveled by the ball in coming to rest.

31. The side of a certain square is 10 inches long. A second square is drawn by connecting the mid-points of the sides of the 1st square; a 3d square is drawn by connecting the mid-points of the sides of the 2d square; etc., without end. Find the sum of the areas of all the squares.

Note 1. If $|r| < 1$, we know that S, of (5) on page 237, is approximately equal to S_n if n is large, and our confidence in this approximation increases as n increases. When n is large, it is decidedly easier to compute S than S_n, and hence it is convenient at times to use S in place of S_n.

32. (I) Find S_{11} for the G.P. $3, \frac{3}{2}, \frac{3}{4}, \cdots$. (II) Find the sum of this progression extended to infinitely many terms.

33. Find S_{139}, approximately, for the G.P. $8 + \frac{8}{3} + \frac{8}{9} + \cdots$.

CHAPTER *15*

LOGARITHMS

164. Logarithms

Logarithms are auxiliary numbers which are exponents, and which permit us to simplify the operations of multiplication, division, raising to powers, and extraction of roots, applied to explicit real numbers.

Previously, we have introduced exponents only under the assumption that they are *rational* numbers. In connection with logarithms, however, when we mention an *exponent,* it may be *any real number, rational* or *irrational.* A logical foundation for the use of irrational exponents is beyond the scope of this text. Hence, without discussion, we shall assume the fact that irrational powers have meaning and that the laws of exponents hold if the exponents involved are real numbers, either *rational* or *irrational,* provided that the base is *positive.*

ILLUSTRATION 1. The student may safely use his intuition in connection with the symbol $10^{\sqrt{2}} = 10^{1.414\cdots}$. Closer and closer approximations to $10^{\sqrt{2}}$ are obtained if the successive decimal approximations to $\sqrt{2}$ are used as exponents.

165. Logarithms to any base

In the following definition, b represents any *positive number,* not 1, and N is any *positive* number.

DEFINITION I. *The logarithm of a number N to the base b is the exponent of the power to which b must be raised to obtain N.*

In other words, if $b^x = N$ then x is the logarithm of N to the base b. To abbreviate "*the logarithm of N to the base b,*" we write "$\log_b N$." Then, by Definition I, the following equations state the same fact, the first equation in exponential form and the second in

logarithmic form:

$$N = b^x \quad and \quad x = \log_b N. \tag{1}$$

ILLUSTRATION 1. If $N = 4^5$, then 5 is the logarithm of N to the base 4.

ILLUSTRATION 2. "$\log_2 64$" is read "*the logarithm of* 64 *to the base* 2":

$$since \quad 64 = 2^6, \quad hence \quad \log_2 64 = 6.$$

ILLUSTRATION 3. Since $\sqrt[3]{5} = 5^{\frac{1}{3}}$, hence $\log_5 \sqrt[3]{5} = \frac{1}{3} = .333\cdots$.

ILLUSTRATION 4. To find $\log_2 \frac{1}{8}$, we express $\frac{1}{8}$ as a power of 2:

$$since \quad \frac{1}{8} = \frac{1}{2^3} = 2^{-3}, \quad hence \quad \log_2 \frac{1}{8} = -3.$$

ILLUSTRATION 5. If $\log_b 16 = 4$, then $b^4 = 16$; $b = \sqrt[4]{16} = 2$.

ILLUSTRATION 6. If $\log_a 2 = -\frac{1}{3}$, then $a^{-\frac{1}{3}} = 2$. Hence,

$$\frac{1}{a^{\frac{1}{3}}} = 2; \quad a^{\frac{1}{3}} = \frac{1}{2}; \quad a = \left(\frac{1}{2}\right)^3 = \frac{1}{8}.$$

For any base b, we have $b^0 = 1$ and $b^1 = b$. Hence,

$$\log_b 1 = 0; \qquad \log_b b = 1. \tag{2}$$

Note 1. We do not use $b = 1$ as a base for logarithms because every power of 1 is 1 and hence no number except 1 could have a logarithm to the base 1.

In the definition of $\log_b N$, we stated that N was a *positive* number. That is, in this book, if we speak of the logarithm of a *number* N we shall mean a *positive* number N. Also, we stated that the base b is *positive*. These agreements were made to avoid meeting imaginary numbers as logarithms. In advanced mathematics it is proved that, if N and b are positive, there exists just *one* real logarithm of N to the base b.

EXERCISE 88

1. Since $27 = 3^3$, what is the logarithm of 27 to the base 3?

2. Since $625 = 5^4$, what is the logarithm of 625 to the base 5?

3. Since $\frac{1}{3} = 3^{-1}$, what is the logarithm of $\frac{1}{3}$ to the base 3?

4. Since $1 = b^0$, what is the logarithm of 1 to the base b?

5. If the logarithm of N to the base 4 is 3, find N.

6. What number has -2 as its logarithm to the base 4?

Find the number N whose logarithm is given.

7. $\log_3 N = 4$.
8. $\log_6 N = 2$.
9. $\log_{10} N = 1$.
10. $\log_{10} N = 2$.
11. $\log_7 N = 0$.
12. $\log_5 N = -1$.
13. $\log_{10} N = -1$.
14. $\log_2 N = -3$.
15. $\log_4 N = -3$.
16. $\log_{10} N = -4$.
17. $\log_4 N = \frac{1}{2}$.
18. $\log_{25} N = \frac{1}{2}$.
19. $\log_8 N = \frac{1}{3}$.
20. $\log_{100} N = 2$.
21. $\log_{100} N = \frac{3}{2}$.

22. Find the logarithm of 125 to the base 5.

23. Find the logarithm of 1,000,000 to the base 10.

Find the specified logarithm.

24. $\log_5 25$.
25. $\log_4 16$.
26. $\log_2 8$.
27. $\log_7 49$.
28. $\log_3 27$.
29. $\log_4 64$.
30. $\log_{10} 1000$.
31. $\log_5 625$.
32. $\log_{25} 5$.
33. $\log_9 3$.
34. $\log_{100} 10$.
35. $\log_{27} 3$.
36. $\log_4 \frac{1}{4}$.
37. $\log_6 \frac{1}{6}$.
38. $\log_2 \frac{1}{4}$.
39. $\log_3 \frac{1}{27}$.

★*Find the value of the unknown letter in the problem.*

40. $\log_b 16 = 2$.
41. $\log_b 125 = 3$.
42. $\log_b 625 = 4$.
43. $\log_b 1000 = 3$.
44. $\log_c 9 = \frac{1}{2}$.
45. $\log_a 4 = -1$.
46. $\log_a 7 = -1$.
47. $\log_b \frac{1}{4} = -2$.
48. $\log_b \frac{25}{9} = 2$.
49. $\log_b 2 = \frac{1}{3}$.
50. $\log_b \frac{1}{25} = -2$.
51. $\log_b .0001 = -2$.
52. $\log_{\frac{1}{4}} 16 = x$.
53. $\log_{\frac{1}{4}} N = -3$.
54. $\log_8 N = -\frac{4}{3}$.

166. Common logarithms

Logarithms to the base 10 are called *common logarithms* and are the most useful variety for computational purposes. Hereafter, unless otherwise stated, when we mention a *logarithm* we shall mean a *common* logarithm. For abbreviation, we shall write $\log N$, instead of $\log_{10} N$, for the common logarithm of N and read $\log N$ as *the* logarithm of N. Then, from the definition of a logarithm, the following equations are equivalent:

$$N = 10^x \quad \textit{and} \quad x = \log N. \tag{1}$$

ILLUSTRATION 1. Since $10{,}000 = 10^4$, hence $\log 10{,}000 = 4$.
Since $\sqrt[3]{10} = 10^{\frac{1}{3}}$, hence $\log \sqrt[3]{10} = .333\cdots$.
Since $.01 = 10^{-2}$, hence $\log .01 = -2$.

We have seen that $\log N$ may be either *positive*, *negative*, or *zero*, depending on the value assigned to N. Also, we notice that $\log N$ is an *integer* when and only when N is an *integral* power of 10.

ILLUSTRATION 2. For future reference, the student should verify the following logarithms by use of the definition of a logarithm.

$N =$	.0001	.001	.01	.1	1	10	100	1000	10,000	100,000
$\log N =$	-4	-3	-2	-1	0	1	2	3	4	5

167. Some properties of logarithms

The following properties hold if the logarithms are taken to *any* base b, but we shall write proofs only for the case where $b = 10$.

I. *The logarithm of a product equals the sum of the logarithms of its factors. For instance,*

$$\log MN = \log M + \log N. \tag{1}$$

ILLUSTRATION 1. If $M = 10^3$ and $N = 10^5$, then $MN = 10^3 10^5 = 10^8$. Also, by the definition of a logarithm, $\log M = 3$, $\log N = 5$, and $\log MN = 8$. Hence, $\log MN = \log M + \log N$ in this special case.

Proof of Property I. 1. Let $\log M = x$ and $\log N = y$. Then,

$$M = 10^x, \qquad N = 10^y, \qquad \text{and} \qquad MN = 10^x 10^y = 10^{x+y}.$$

2. Since $MN \doteq 10^{x+y}$, hence $\log MN = x + y = \log M + \log N$.

II. *The logarithm of a quotient equals the logarithm of the dividend minus the logarithm of the divisor. That is,*

$$\log \frac{M}{N} = \log M - \log N. \tag{2}$$

Proof. 1. Let $\log M = x$ and $\log N = y$. Then,

$$\frac{M}{N} = \frac{10^x}{10^y} = 10^{x-y}. \qquad \text{(By a law of exponents)}$$

2. Hence, $\log \dfrac{M}{N} = x - y = \log M - \log N$.

ILLUSTRATION 2. If we are given $\log 3 = .4771$, then

$$\log 300 = \log 3(100) = \log 3 + \log 100 = .4771 + 2 = 2.4771;$$
$$\log .003 = \log \tfrac{3}{1000} = \log 3 - \log 1000 = .4771 - 3 = -2.5229.$$

EXERCISE 89

Express each number as a product or quotient, and find its logarithm by use of the given logarithms and the logarithms of integral powers of 10.

$$\log 2 = .3010;\ \log 3 = .4771;\ \log 7 = .8451;\ \log 17 = 1.2304.$$

1. 6. **2.** 21. **3.** 34. **4.** 51. **5.** 30. **6.** 70.

7. 1700. **8.** 2000. **9.** 42. **10.** $\frac{3}{2}$. **11.** $\frac{7}{3}$. **12.** $\frac{17}{7}$.

13. $\frac{17}{3}$. **14.** $\frac{3}{7}$. **15.** $\frac{7}{17}$. **16.** $\frac{7}{100}$. **17.** $\frac{3}{10}$. **18.** .17.

19. .007. **20.** .0003. **21.** .017. **22.** .0119. **23.** .042. **24.** 10.2.

168. Characteristic and mantissa

Every number, and hence *every logarithm*, can be written as the sum of *an integer and a decimal fraction* which is *positive or zero* and *less than* 1. When $\log N$ is written in this way, we call the integer the *characteristic* and the fraction the *mantissa* of $\log N$.

$$\textbf{log } N = \textbf{(an integer)} + \textbf{(a fraction, } \geqq 0, < 1\textbf{)};$$
$$\textbf{log } N = \textbf{characteristic} + \textbf{mantissa}. \qquad (1)$$

ILLUSTRATION 1. If $\log N = 4.6832 = 4 + .6832$, then .6832 is the mantissa and 4 is the characteristic of $\log N$.

ILLUSTRATION 2. If $\log N = -3.75$, then $\log N$ lies between -4 and -3. Hence, $\log N = -4 +$ (*a fraction*). To find the fraction, subtract: $4 - 3.75 = .25$. Hence, $\log N = -3.75 = -4 + .25$.

ILLUSTRATION 3. The following logarithms were obtained by later methods. The student should verify the three columns at the right.

	LOGARITHM	CHARACTERISTIC	MANTISSA
log 300 = 2.4771	= 2 + .4771	2	.4771
log 50 = 1.6990	= 1 + .6990	1	.6990
log .001 = − 3	= − 3 + .0000	− 3	.0000
log 6.5 = 0.8129	= 0 + .8129	0	.8129
log .0385 = − 1.4145	= − 2 + .5855	− 2	.5855
log .005 = − 2.3010	= − 3 + .6990	− 3	.6990

169. Properties of the characteristic and mantissa

ILLUSTRATION 1. All numbers whose logarithms are given below have the same significant digits (3, 8, 0, 4). To obtain the logarithms, log 3.804 was first found from a table to be discussed later; the other logarithms were then obtained by the use of Properties I and II.

$$
\begin{array}{llll}
\log 380.4 & = \log 100(3.804) & = \log 100 + \log 3.804 & = \quad 2 + \mathbf{.5802}; \\
\log 38.04 & = \log 10(3.804) & = \log 10 + \log 3.804 & = \quad 1 + \mathbf{.5802}; \\
\log 3.804 & = .5802 & & = \quad 0 + \mathbf{.5802}; \\
\log .3804 & = \log \dfrac{3.804}{10} & = \log 3.804 - \log 10 & = -1 + \mathbf{.5802}; \\
\log .03804 & = \log \dfrac{3.804}{100} & = \log 3.804 - \log 100 & = -2 + \mathbf{.5802}.
\end{array}
$$

Similarly, if N is *any* number whose significant digits are (3, 8, 0, 4), then N equals 3.804 multiplied, or else divided, by a positive integral power of 10; hence, it follows as before that .5802 is the mantissa of log N.

In Illustration 1, the characteristic of log 380.4 is 2, of log 38.04 is 1, etc. These facts could have been learned as follows.

ILLUSTRATION 2. To find the characteristic of log 380.4, notice the two successive integral powers of 10 between which 380.4 lies:

$$100 < 380.4 < 1000.$$

Hence, $\quad \log 100 < \log 380.4 < \log 1000; \quad$ or, $\quad 2 < \log 380.4 < 3.$

Therefore, $\log 380.4 = 2 + (\text{a fraction}, > 0, < 1)$; or, by definition, the characteristic of log 380.4 is 2.

In Illustration 1 we met special cases of the following theorems.

THEOREM I. *The mantissa of* log N *depends only on the sequence of significant digits in* N. *That is,* **if two numbers differ only in the position of the decimal point, their logarithms have the same mantissa.**

THEOREM II. **When $N > 1$,** *the characteristic of* log N *is an integer, positive or zero, which is* **one less than the number of digits in N to the left of the decimal point.**

THEOREM III. **If $N < 1$,** *the characteristic of* log N *is a negative integer;* **if the first significant digit of N is in the kth decimal place, then $-k$ is the characteristic of log N.**

ILLUSTRATION 3. By use of Theorems II and III, we find the characteristic of $\log N$ by merely inspecting N. Thus, by Theorem III, the characteristic of $\log .00039$ is -4 because "3" is in the 4th decimal place. By Theorem II, the characteristic of $\log 1578.6$ is 3.

170. Standard form for a negative logarithm

Hereafter, for convenience in computation, **if the characteristic of $\log N$ is negative, $-k$, change it to the equivalent value**

$$[(10 - k) - 10], \quad \text{or} \quad [(20 - k) - 20], \text{ etc.}$$

ILLUSTRATION 1. Given that $\log .000843 = -4 + .9258$, we write

$$\log .000843 = -4 + .9258 = (6 - 10) + .9258 = 6.9258 - 10.$$

The characteristics of the following logarithms are obtained by use of Theorem III; the mantissas are identical, by Theorem I.

1ST SIGNIF. DIGIT IN	ILLUSTRATION	LOG N	STANDARD FORM
1st *decimal place*	$N = .843$	$-1 + .9258$	$= 9.9258 - 10$
2d *decimal place*	$N = .0843$	$-2 + .9258$	$= 8.9258 - 10$
6th *decimal place*	$N = .00000843$	$-6 + .9258$	$= 4.9258 - 10$

171. Four-place table of logarithms

Mantissas can be computed by use of advanced mathematics, and, except in special cases, are endless decimal fractions. Computed mantissas are found in *tables of logarithms*, also called *tables of mantissas*.

ILLUSTRATION 1. The mantissa for $\log 10705$ is $.029586671630457$, to fifteen decimal places.

Table II gives the mantissa of $\log N$ correct to four decimal places if N has at most three significant digits; a decimal point is understood in front of each mantissa in the table. If N lies between 1 and 10, the characteristic of $\log N$ is zero so that $\log N$ is *the same as its mantissa*. Hence, a four-place table of *mantissas* is also a table of the *actual logarithms of all numbers with at most three significant digits*, from $N = 1.00$ to $N = 10.00$. In case N is less than 1 or greater than 10, we must supply the characteristic of $\log N$ by use of Theorems II and III besides obtaining the mantissa of $\log N$ by use of Table II.

EXAMPLE 1. Find log .0316 from Table II.

SOLUTION. 1. *To obtain the mantissa:* find "31" in the column headed N in the table; in the row for "31," read the entry in the column headed "6." The mantissa is .4997.

2. By Theorem III, the characteristic of log .0316 is -2, or $(8-10)$:

$$\log .0316 = -2 + .4997 = 8.4997 - 10.$$

ILLUSTRATION 2. From Table II and Theorem II, $\log 31{,}600 = 4.4997$.

EXAMPLE 2. Find N if $\log N = 7.6064 - 10$.

SOLUTION. 1. *To find the significant digits of N:* the mantissa of $\log N$ is .6064; this is found in Table II as the mantissa for the digits "404."

2. *To locate the decimal point in N:* the characteristic of $\log N$ is $(7-10)$ or -3; hence, by Theorem III, $N = .00404$.

ILLUSTRATION 3. If $\log N = 3.6064$, the characteristic is 3 and, by Theorem II, N has 4 figures to the left of the decimal point: the mantissa is the same as in Example 2. Hence, $N = 4040$.

DEFINITION I. *A number N is called the* **antilogarithm** *of L in case* $\log N = L$, *and for abbreviation we write* $N =$ **antilog** L.

ILLUSTRATION 4. Since $\log 1000 = 3$, hence $1000 = \text{antilog } 3$.

ILLUSTRATION 5. In Example 2 we found *antilog* $(7.6064 - 10)$.

EXERCISE 90

In Problems 1 *to* 8, *each number is the logarithm of some number N. State the characteristic and the mantissa of* $\log N$.

1. 2.9356. **2.** 15.2162. **3.** -1.300. **4.** $-2 + .3561$.

5. 3.5473. **6.** $7.2356 - 10$. **7.** -5.675. **8.** $5.1942 - 10$.

Write the following negative logarithms in standard form.

9. $-1 + .2562$. **10.** $.3267 - 3$. **11.** $.4932 - 6$. **12.** -3.4675.

State the characteristic of the logarithm of each number.

13. 637,500. **14.** 368. **15.** .000673. **16.** .00897. **17.** .000007.

Use Table II *to find the four-place logarithm of the number.*

18. 65.4. **19.** 43.2. **20.** 178. **21.** .00785. **22.** .0346.

23. 9.46. **24.** 6530. **25.** 17,800. **26.** .00005. **27.** .086.

28. .000358. **29.** 101,000. **30.** .00089. **31.** 157,000. **32.** .0000002.

Find the antilogarithm of the given logarithm by use of Table II.

33. 2.3856. **34.** 3.3927. **35.** 3.6684. **36.** 1.8785. **37.** 0.1553.

38. 2.1461. **39.** 1.8692. **40.** 0.9727. **41.** 2.4800. **42.** 0.5611.

43. 7.7701 − 10. **44.** 9.8041 − 10. **45.** 8.9823 − 10.

46. 4.8915 − 10. **47.** 4.9542 − 10. **48.** 2.9340 − 10.

49. 9.4216 − 10. **50.** 8.7284 − 10. **51.** − 2.3010.

172. Interpolation for a mantissa

Interpolation in a table of mantissas is based on the assumption that, *for small changes in* N, *the corresponding changes in* log N *are proportional to the changes in* N. This **principle of proportional parts** is merely an approximation to the truth but leads to results which are sufficiently accurate for our purposes.

We agree that, whenever a mantissa is found by interpolation from a table, we shall express the result *only to the number of decimal places given in table entries.* Also, in finding N by interpolation in a table of mantissas when log N is given, we agree to specify just **four** or just **five** significant digits according as we are using a **four-place** or a **five-place** table. No greater refinement in the result is justified because the unavoidable error, which may arise, frequently will be as large as 1 unit in the last significant digit which we have agreed to specify, although the error is rarely larger.

EXAMPLE 1. Find log 13.86 by interpolation in Table II.

SOLUTION. 1. We notice that $13.80 < 13.86 < 13.90$. Hence, by the principle of proportional parts, we assume that, since 13.86 is $\frac{6}{10}$ of the way from 13.80 to 13.90,

$$\log 13.86 \textit{ is } \tfrac{6}{10} \textit{ of the way from } \log 13.80 \textit{ to } \log 13.90, \textit{ or}$$
$$\log 13.86 = \log 13.80 + .6(\log 13.90 - \log 13.80).$$

2. Each logarithm below has 1 for its characteristic, by Theorem II.

$$\begin{array}{lll} \textit{From table:} & \log 13.80 = 1.1399 & \\ & \log 13.86 = \quad ? & \left.\right\} 31 \\ \textit{From table:} & \log 13.90 = 1.1430 & \end{array}$$

Tabular difference *is* $.1430 - .1399 = .0031.$ $.6(.0031) = .00186$, or $.0019$.

$$\log 13.86 = 1.1399 + .6(.0031) = 1.1399 + .0019 = 1.1418.$$

Comment. We found $.6(31) = 18.6$ by use of the table headed "31" under the *column of proportional parts* in Table II.

Illustration 1. To find log .002914:

$$10\left[\begin{array}{l} 4\left[\begin{array}{l} 2910:\ \textit{mantissa is } .4639 \\ 2914:\ \textit{mantissa is } \ ? \end{array}\right]x \\ \quad 2920:\ \textit{mantissa is } .4654 \end{array}\right]15$$

Tabular difference *is* $.4654 - .4639 = .0015.$ $x = .4(15) = 6.$

Hence, the mantissa for 2914 *is* $.4639 + .0006 = .4645.$

Hence, by Theorem III, $\log .002914 = -3 + .4645 = 7.4645 - 10.$

Example 2. Find N if $\log N = 1.6187$.

Solution. 1. The mantissa .6187 is not in Table II but lies between the consecutive entries .6180 and .6191, the mantissas for 415 and 416.

2. Since .6187 is $\frac{7}{11}$ of the way from .6180 to .6191, we assume that N is $\frac{7}{11}$ of the way from 41.50 to 41.60.

$$11\left[\begin{array}{l} 7\left[\begin{array}{l} 1.6180 = \log 41.50 \\ 1.6187 = \log N \end{array}\right]x \\ \quad 1.6191 = \log 41.60 \end{array}\right].10$$

$41.60 - 41.50 = .10$ $\quad x = \frac{7}{11}(.10) = .064,$ *or approximately* .06.

$$N = 41.50 + \tfrac{7}{11}(.10) = 41.50 + .06 = 41.56.$$

Illustration 2. To find N if $\log N = 6.1053 - 10$:

$$34\left[\begin{array}{l} 15\left[\begin{array}{l} .1038,\ \textit{mantissa for } 1270 \\ .1053,\ \textit{mantissa for } \ ? \end{array}\right]x \\ \quad .1072,\ \textit{mantissa for } 1280 \end{array}\right]10$$

$\frac{15}{34} = .4.$ *Hence,* $x = .4(10) = 4.$ $\quad 1270 + 4 = 1274.$

Hence, .1053 *is the mantissa for* 1274 *and* $N = .0001274.$

Comment. We obtain $\frac{15}{34} = .4$ by inspection of the tenths of 34 in the columns of proportional parts. We read

$$13.6 = .4(34) \quad \text{or} \quad \frac{13.6}{34} = .4, \quad \text{and} \quad \frac{17}{34} = .5.$$

Since 15 is nearer to 13.6 than to 17, hence $\frac{15}{34}$ is nearer to .4 than to .5.

Note 1. When interpolating in a table of mantissas, if there is equal reason for choosing either of two successive digits, for uniformity we agree to make that choice which gives an **even digit** in the last significant place of the *final result* of the interpolation.

173. Scientific notation for a number

Any positive number N can be written in the form

$$N = P(10^k), \tag{1}$$

where P is a number greater than or equal to 1 but less than 10, and k is an integer, either zero or positive or negative. We refer to the right-hand side of (1) as the *scientific notation* for N.

ILLUSTRATION 1. $5{,}832{,}900 = 5.8329(10^6).$

$$.00000058329 = 5.8329(.0000001) = 5.8329(10^{-7}).$$

The scientific notation gives a brief and easily appreciated form for writing very large or very small numbers.

ILLUSTRATION 2. The nucleus of an atom has a diameter which is estimated as less than $3(10^{-12})$ centimeters. The mean distance from the sun to the outermost planet Pluto is approximately 3.67×10^9 miles. One light-year, the distance which light will travel in one year in interstellar space, is approximately 6×10^{12} miles.

In equation 1, N and P have the *same significant digits* because the factor 10^k merely alters the position of the decimal point to change P into N. Hence, the scientific notation is very useful in writing a number N, particularly if it is very large, when we wish to show how many digits in N are significant. This feature was referred to in Section 48, page 53, for the case where k of (1) is positive.

ILLUSTRATION 3. If 68,820,000 is the approximate value of some quantity and if just five digits are significant, this is not indicated by the usual form of the number. We write it as $6.8820(10^7)$ to show that one of the zeros is significant.

ILLUSTRATION 4. If $N = 1.352(10^8)$, then, by Property I, page 243,

$$\log N = \log 1.352 + \log 10^8 = 0.1309 + 8 = 8.1309.$$

Thus, 8 is the characteristic and log 1.352 is the mantissa of $\log N$.

Consider any number N, where $N = P(10^k)$ where k is an integer and $1 \leqq P < 10$. Then k is the characteristic and $\log P$ is the mantissa of $\log N$, because

$$\log N = \log P + \log 10^k = \log P + k,$$

where $0 \leqq \log P < 1$, since $1 \leqq P < 10$.

ILLUSTRATION 5. If $\log N = 9.7419$, and if we use the form $N = P(10^k)$, we have $k = 9$ and $\log P = .7419$:

$$P = 5.520 \quad \text{and} \quad N = 5.520(10^9). \qquad \text{(Four digits significant.)}$$

Note 1. Besides common logarithms, the only other variety used appreciably is the system of **natural,** or **Naperian logarithms,** for which the base is a certain irrational number denoted by e where $e = \mathbf{2.71828} \cdots$. Natural logarithms are useful for theoretical purposes.

Note 2. Logarithms were invented by a Scotchman, JOHN NAPIER, Baron of Merchiston (1550–1617). His original logarithms were not the same as those now called Naperian logarithms, in his honor. Common logarithms, also called Briggs logarithms, were invented by an Englishman, HENRY BRIGGS (1556–1631), who was aided by Napier.

EXERCISE 91

Find the four-place logarithm of each number from Table II.

1. 1826.	**2.** 25.63.	**3.** 532.2.	**4.** 12.67.
5. 35.94.	**6.** 1.293.	**7.** .3013.	**8.** .4213.
9. .5627.	**10.** .03147.	**11.** .01563.	**12.** .001139.
13. 90,090.	**14.** 203,500.	**15.** .001439.	**16.** .05626.
17. $1.233(10^{-4})$.	**18.** $1.417(10^{6})$.	**19.** $3.126(10^{3})$.	**20.** $2.438(10^{-2})$.

Find the antilogarithm of each four-place logarithm from Table II.

21. 3.2367.	**22.** 7.1247 − 10.	**23.** 6.1640.	**24.** 8.9935 − 10.
25. 3.1395.	**26.** 2.9276.	**27.** 1.6016.	**28.** 0.4906.
29. 6.3350 − 10.	**30.** 4.1436 − 10.	**31.** 9.6715 − 10.	**32.** 8.0255 − 10.
33. 8.8862.	**34.** 2.1952.	**35.** 0.0130.	**36.** 5.5511.
37. 5.9885 − 10.	**38.** 8.3358 − 20.	**39.** 9.6270 − 10.	**40.** 6.4228.

174. Computation of products and quotients

Unless otherwise specified, we shall assume that the data of any given problem are *exact.* Under this assumption, the accuracy of a product, quotient, or power computed by use of logarithms depends on the number of places in the table being used. The result is frequently subject to an unavoidable error which usually is at most a few units in the last significant place given by interpolation. Hence, usually we must compute with at least *five-place* logarithms to obtain *four-place accuracy,* and with at least *four-place* logarithms to obtain *three-place accuracy.* As a general custom, in any result we shall give *all digits obtainable by interpolation* in the specified table.

EXAMPLE 1. Compute .0631(7.208)(.5127) by use of Table II.

SOLUTION. Let P represent the product. By Property I, we obtain $\log P$ by adding the logarithms of the factors. We obtain the logarithms of the factors from Table II, add to obtain $\log P$, and then finally obtain P from Table II. The computing form, given in blackface type, was made up completely as *the first step in the solution.*

$$\begin{aligned}
\mathbf{\log .0631} &= 8.8000 - 10 && \text{(Table II)}\\
\mathbf{\log 7.208} &= 0.8578 && \text{(Table II)}\\
\mathbf{\log .5127} &= 9.7099 - 10 && \text{(Table II)}\\
\hline
(\textit{add})\ \mathbf{\log P} &= 19.3677 - 20 = 9.3677 - 10.\\
\mathbf{Hence,}\ \mathbf{P} &= .2332. \qquad [= \text{antilog}\ (9.3677 - 10),\ \text{Table II}]
\end{aligned}$$

EXAMPLE 2. Compute $q = \dfrac{431.91}{15.6873}$.

SOLUTION. 1. By Property II, $\log q$ equals *the logarithm of the numerator minus the logarithm of the denominator.*

2. Before computing, we *round off* each given number to *four* significant digits because we are using a four-place table. For instance, 15.6873 becomes 15.69.

$$\begin{aligned}
\mathbf{\log 431.9} &= 2.6354 && \text{(Table II)}\\
(-)\ \mathbf{\log 15.69} &= 1.1956 && \text{(Table II)}\\
\hline
\mathbf{\log q} &= 1.4398. \quad \mathbf{Hence,}\ \mathbf{q} = 27.53. && \text{(Table II)}
\end{aligned}$$

EXAMPLE 3. Compute $q = \dfrac{257}{8956}$.

SOLUTION. We employ Property II.

$$\begin{aligned}
\mathbf{\log 257} &= 2.4099 = 12.4099 - 10\\
(-)\ \mathbf{\log 8956} &= 3.9521 = 3.9521\\
\hline
\mathbf{\log q} &= \ ??? \ = 8.4578 - 10. \quad \mathbf{Hence,}\ \mathbf{q} = .02869.
\end{aligned}$$

Comment. When we first tried to subtract log 8956 from log 257, we saw that the result would be *negative* because log 8956 is *greater* than log 257. In order that $\log q$ should appear immediately in the *standard form for a negative logarithm,* we changed log 257 by adding 10 and then subtracting 10 to compensate for the first change. Actually,

$$\log q = 2.4099 - 3.9521 = -1.5422 = 8.4578 - 10.$$

Whenever it is necessary to subtract a larger logarithm from a smaller one in computing a quotient, add 10 to the characteristic of the smaller logarithm and then subtract 10 to compensate for the change.

EXAMPLE 4. Compute $q = \dfrac{(4.803)(269.9)(1.636)}{(7880)(253.6)}$.

INCOMPLETE SOLUTION. First make a **computing form.**

$$\begin{array}{rl} (+)\left\{\begin{array}{l}\log 4.803 = \\ \log 269.9 = \\ \log 1.636 = \end{array}\right. & \\ \hline \log \text{numer.} = & \\ (-)\ \log \text{denom.} = & \\ \hline \log q = & \end{array} \qquad \begin{array}{l} (+)\left\{\begin{array}{l}\log 7880 \ = \\ \log 253.6 = \end{array}\right. \\ \hline \log \text{denom.} = \\ \\ \\ \text{Hence, } q = \end{array}$$

EXAMPLE 5. Compute the reciprocal of 189 by use of Table II.

SOLUTION. Let $R = \dfrac{1}{189}$.

$$\begin{array}{rcl} \log 1 = 0.0000 &=& 10.0000 - 10 \\ (-)\ \log 189 = 2.2765 &=& 2.2765 \\ \hline \log R = \quad ? &=& 7.7235 - 10. \end{array}$$

Hence, $R = .005290$.

Comment. In writing any approximate value, it is essential to indicate all final zeros which are significant. Hence, in writing $R = .005290$ in Example 5, the final zero was essential. It would have been wrong to write $R = .00529$, because this would not show that we had reasonably accurate information concerning the next digit, zero.

Note 1. Before finding the *four*-place log N if N has more than four significant digits, round off N to *four* significant digits.

EXERCISE 92

Compute by use of four-place logarithms.

1. $31.57 \times .789$. **2.** $.8475 \times .0937$. **3.** $925.618 \times .000217$.

4. $925.6 \times .137$. **5.** $.0179 \times .35641$. **6.** $3.41379 \times .0142$.

7. $(-84.75)(.00368)(.02458)$. **8.** $(-16.8)(136.943)(.00038)$.

HINT. Only *positive* numbers have *real* logarithms. First compute as if all factors were positive; then determine the sign by inspection.

9. $\dfrac{675}{13.21}$. **10.** $\dfrac{728.72}{895}$. **11.** $\dfrac{.0894}{.6358}$. **12.** $\dfrac{1}{325.932}$.

13. $\dfrac{568.5}{23.14}$. **14.** $\dfrac{753.166}{9273.8}$. **15.** $\dfrac{.0421}{.53908}$. **16.** $\dfrac{1}{100{,}935}$.

17. $\dfrac{16.083 \times 256}{47 + .0158}$. **18.** $\dfrac{9.32 \times 531}{.8319 \times .5685}$. **19.** $\dfrac{1}{.53819 \times .0673}$.

20. $\dfrac{.42173 \times .217}{.3852 \times .956}$. **21.** $\dfrac{5.4171 \times .429}{18.1167 \times 37}$. **22.** $\dfrac{1}{.00073 \times .965}$.

23. $\dfrac{(-.29)(.038)(-.0065)}{(-1006.332)(2.71)}$. **24.** $\dfrac{(5.6)(-3.9078)(-.00031)}{(132)(-1.93)}$.

Compute the reciprocal of the number.

25. 63283. **26.** .00382. **27.** .02567. **28.** .0683(.52831).

29. (*a*) Compute 652(735); (*b*) compute (log 652)(log 735).

30. (*a*) Compute .351 ÷ 625; (*b*) compute (log .351) ÷ (log 625).

★175. Cologarithms *

The logarithm of the *reciprocal* of N is called the *cologarithm* of N and is written **colog** N. Since log 1 = 0,

$$\textbf{colog } N = \textbf{log } \frac{1}{N} = 0 - \textbf{log } N. \tag{1}$$

ILLUSTRATION 1. Colog .031 = log $\dfrac{1}{.031}$:

$$\begin{array}{rl} \textbf{log 1} &= 10.0000 - 10 \\ (-)\ \textbf{log .031} &= \underline{\ \ 8.4914 - 10} \\ \textbf{colog .031} &= \ \ 1.5086. \end{array}$$

The positive part of colog N can be quickly obtained by inspection of log N: *subtract each digit (except the last) in the positive part of log N from* 9, *and subtract the last digit from* 10.

EXAMPLE 1. Compute $q = \dfrac{16.083 \times 256}{47 \times .0158}$ by use of cologarithms.

SOLUTION. To *divide* by N is the same as to *multiply* by $1/N$. Hence, instead of *subtracting the logarithm* of each factor of the denominator, we *add the cologarithm* of the factor:

$$q = \frac{16.083 \times 256}{47 \times .0158} = (16.083 \times 256)\left(\frac{1}{47}\right)\left(\frac{1}{.0158}\right).$$

$$\begin{array}{lrl} & \textbf{log 16.08} &= \ \ 1.2063 \\ & \textbf{log 256} &= \ \ 2.4082 \\ \textbf{log 47} = 1.6721;\ \textbf{hence,} & \textbf{colog 47} &= \ \ 8.3279 - 10 \\ \textbf{log .0158} = 8.1987 - 10;\ \textbf{hence,} & \textbf{colog .0158} &= \ \ \underline{1.8013} \\ q = 5542. \longleftarrow & (\textit{add})\ \textbf{log } q &= 13.7437 - 10 = 3.7437. \end{array}$$

* The instructor may wish to direct occasional use of cologarithms.

176. Computation of powers and roots

We establish the following property of logarithms as an aid to computing powers.

III. *The logarithm of the* kth *power of a number* N *equals* k *times the logarithm of* N:

$$\log N^k = k \log N. \tag{1}$$

Proof. Let $x = \log N$. Then, by the definition of a logarithm, $N = 10^x$. Hence,

$$N^k = (10^x)^k = 10^{kx}. \qquad \text{(A law of exponents)}$$

Therefore, by the definition of a logarithm,

$$\log N^k = kx = k \log N. \qquad (\text{Using } x = \log N)$$

ILLUSTRATION 1. $\quad \operatorname{Log} 7^5 = 5 \log 7. \quad \operatorname{Log} N^3 = 3 \log N.$

EXAMPLE 1. Compute $(.3156)^4$.

SOLUTION. By Property III,

$$\log (.3156)^4 = 4 \log (.3156) = 4(9.4991 - 10).$$
$$\log (.3156)^4 = 37.9964 - 40 = 7.9964 - 10.$$
$$\text{Therefore, } (.3156)^4 = .009918.$$

Recall that *any root of a number is expressible as a fractional power.* Hence, as a special case of Property III we obtain

IV. $$\log \sqrt[h]{N} = \frac{\log N}{h}.$$

Proof. Since $\sqrt[h]{N} = N^{\frac{1}{h}}$, we use Property III with $k = \dfrac{1}{h}$:

$$\log \sqrt[h]{N} = \log N^{\frac{1}{h}} = \frac{1}{h} \log N.$$

ILLUSTRATION 2. Since $\sqrt{N} = N^{\frac{1}{2}}$ and $\sqrt[3]{N} = N^{\frac{1}{3}}$,

$$\log \sqrt{N} = \frac{1}{2} \log N; \qquad \log \sqrt[3]{N} = \frac{1}{3} \log N.$$

EXAMPLE 2. Compute $\sqrt[6]{.08351}$.

SOLUTION. By Property IV, $\log \sqrt[6]{N} = \frac{1}{6} \log N$. Hence,

$$\log \sqrt[6]{.08351} = \frac{\log .08351}{6} = \frac{8.9218 - 10}{6};$$

$$\log \sqrt[6]{.08351} = \frac{58.9218 - 60}{6} = 9.8203 - 10. \qquad (2)$$

$$\text{Therefore, } \sqrt[6]{.08351} = .6611.$$

Comment. Before dividing a negative logarithm by a positive integer, usually it is best to write the logarithm in such a way that *the negative part after division, will be* -10. Thus, in (2), we altered $(8.9218 - 10)$ by *subtracting* 50 from -10 to make it -60, and by *adding* 50 to 8.9218 to compensate for the subtraction; the result after division by 6 is in the standard form for a negative logarithm.

Example 3. Compute $q = \left(\dfrac{(.5831)^3}{65.3\sqrt{146}}\right)^{\frac{2}{5}}$.

Solution. 1. Let F represent the fraction. Then $\log q = \frac{2}{5} \log F$.

2. Notice that $\log (.5831)^3 = 3 \log .5831$; $\log \sqrt{146} = \frac{1}{2} \log 146$.

$$\begin{aligned} \log .5831 &= 9.7658 - 10 \\ \log 146 &= 2.1644 \\ \hline 3 \log .5831 &= 9.2974 - 10 \\ (-) \log \text{denom.} &= 2.8971 \\ \hline \end{aligned} \qquad\qquad (+)\left\{\begin{aligned} \log 65.3 &= 1.8149 \\ \tfrac{1}{2} \log 146 &= 1.0822 \end{aligned}\right. \\ \hline \log \text{denom.} = 2.8971.$$

$$\log F = 6.4003 - 10; \; 2 \log F = 2.8006 - 10 = 42.8006 - 50.$$

$$\log q = \frac{2 \log F}{5} = \frac{42.8006 - 50}{5} = 8.5601 - 10.$$

Hence, $q = .03632$.

EXERCISE 93

Compute by use of four-place logarithms.

1. $(17.5)^3$. 2. $(3.1279)^4$. 3. $(.837)^5$. 4. $(.0315)^3$.

5. $\sqrt{1.09}$. 6. $\sqrt[3]{2795}$. 7. $\sqrt[3]{.857}$. 8. $\sqrt[4]{.03107}$.

9. $(1.04)^7$. 10. $(10{,}000)^{\frac{1}{3}}$. 11. $(.0797)^{\frac{1}{5}}$. 12. $(.0138273)^{\frac{1}{2}}$.

13. $(700{,}928)^{\frac{1}{4}}$. 14. $\sqrt[6]{1.045}$. 15. $\sqrt[5]{.0001}$. 16. $\sqrt[3]{.00001}$.

17. $(-1.03)^{\frac{1}{3}}$. 18. $(-1796)^{\frac{1}{5}}$. 19. $(357)^{\frac{2}{3}}$. 20. $(-.00831)^3$.

21. $(143.54)^{\frac{3}{4}}$. 22. $\sqrt{\sqrt[5]{.847}}$. 23. $(-.0057)^{\frac{2}{5}}$. 24. $(157)^{-3}$.

Hint for Problem 24. Recall that $(157)^{-3} = 1/(157^3)$.

25. $(13.67)^{\frac{2}{7}}$. 26. $(3.035)^{-4}$. 27. $(.98)^{-2}$. 28. $(.831447)^{-5}$.

29. $(1.03)^{-5}$. **30.** $(1.05)^{-6}$. **31.** $(1.04)^{100}$. **32.** $(1.04)^{-50}$.

Note 1. Given the seven-place log 1.04 = 0.0170333, compute Problems 31 and 32 and compare with the less accurate former answers.

Compute by use of four-place logarithms.

33. $.958(12.167)^2$. **34.** $10^{1.65}\sqrt{8.265}$. **35.** $10^{2.36}\sqrt{.88147}$. **36.** $(25.3)^2\sqrt[3]{.093}$.

37. $\dfrac{56.3 \times 4.317}{21.4\sqrt{521.923}}$. **38.** $\dfrac{(25.73)(152)^3}{1893.32}$. **39.** $\dfrac{.0198}{(3.82616)^2}$.

40. $\dfrac{758.32}{(46.3)^3}$. **41.** $\sqrt{\dfrac{89.1}{163 \times .62}}$. **42.** $\sqrt[3]{\dfrac{47.5317}{.031 \times .964}}$.

43. $\dfrac{10^{-.36}\sqrt{.78}}{(.983174)^2}$. **44.** $\dfrac{10^{-1.42}\sqrt{.387}}{(57)(8.64)^2}$. **45.** $\dfrac{\sqrt[3]{-463.19}}{\sqrt{16.3144}}$.

46. $\sqrt[3]{\dfrac{(-316)(.198)}{.756392}}$. **47.** $\left(\dfrac{54.2\sqrt{1.89}}{.157386}\right)^{\frac{1}{2}}$. **48.** $\left(\dfrac{5731.84}{14.2\sqrt{.896}}\right)^{\frac{3}{4}}$.

Note 2. Observe that no property of logarithms is available to simplify the computation of a *sum*. Use logarithms below wherever possible.

49. $\dfrac{(35.6)^2 + 89.532}{\sqrt{57} + 2.513}$. **50.** $\dfrac{\sqrt[3]{45} - 364.1}{(.9873)^2 + 16.3}$. **51.** $\dfrac{\sqrt{23} - \sqrt{134.91}}{453 \times .110173}$.

52. $\dfrac{(1.03)^5 + 1}{(1.03)^{\frac{1}{2}} + 1}$. **53.** $\dfrac{\log 86}{\log 53.8}$. **54.** $\dfrac{\log 567 - 20}{\log 235}$.

55. $(2.67)^{1.62}$. **56.** $(53.17)^{.84}$. **57.** $(59.2)^{-.43}$. **58.** $(.065)^{.532}$.

HINT for Problem 57. $-.43 \log 59.2 = -.7621 = 9.2379 - 10$.

59. Compute (*a*) $(\text{antilog } 2.6731)^2$; (*b*) $[\text{antilog }(-1.4973)]^2$.

DEFINITION I. *The* **geometric mean** *of n numbers is defined as the* nth *root of the product of the numbers.* Thus, the geometric mean of M, N, P, Q, and R is $\sqrt[5]{MNPQR}$.

In each problem, find the geometric mean of the given numbers.

60. 138; 395; 426; 537; 612. **61.** .00138; .19276; .08356; .0131.

If a, b, and c are the three sides of a triangle, it is proved in trigonometry that A, the area of the triangle, is given by

$$A = \sqrt{S(S-a)(S-b)(S-c)}, \text{ where } S = \tfrac{1}{2}(a+b+c).$$

Find the area of a triangle whose sides are as follows.

62. 375.40; 141.37; 451.20. **63.** .089312; .0739168; .024853.

The time t in seconds for one oscillation of a simple pendulum whose length is l centimeters, is given by $t = \pi\sqrt{\frac{l}{g}}$, *where* $g = 980$ *and* $\pi = 3.1416$.

64. (*a*) Find the time for one oscillation of a simple pendulum .985 centimeters long. (*b*) Find l if the time for one oscillation is 3.75 seconds.

65. Let d be the diameter in inches of a short solid circular steel shaft which is designed to transmit safely H horse power when revolving at R revolutions per minute. A safe value for d is

$$d = \sqrt[3]{\frac{38H}{R}}.$$

Find the number of horse power which can be safely transmitted at 1150 revolutions per minute if $d = 1.9834$.

66. The weight w, in pounds of steam per second, which will flow through a hole whose cross-section area is A square inches, if the steam approaches the hole under a pressure of P pounds per square inch, is approximately $w = .0165AP^{.97}$. How much steam at a pressure of 83.85 pounds per square inch will flow through a hole 12.369 inches in diameter?

★177. Exponential and logarithmic equations

A *logarithmic equation* is one in which there appears the logarithm of some expression involving the unknown quantity.

EXAMPLE 1. Solve for x: $\qquad \log x + \log \frac{2x}{5} = 6.$

SOLUTION. By use of Properties I and II of page 243,

$$\log x + \log 2 + \log x - \log 5 = 6.$$

$$2 \log x = 6 + \log 5 - \log 2 = 6.3980. \qquad \text{(Table II)}$$

$$\log x = 3.1990; \; x = \text{antilog } 3.1990 = 1581. \qquad \text{(Table II)}$$

An equation where the unknown quantity appears in an exponent is called an *exponential equation*. Sometimes, an exponential equation can be solved by taking the logarithm of each member.

EXAMPLE 2. Solve $16^x = 74$.

SOLUTION. Equate the logarithms of the two sides: $\quad x \log 16 = \log 74;$

$$x = \frac{\log 74}{\log 16} = \frac{1.8692}{1.2041}. \qquad \begin{array}{r} \log 1.869 = 0.2716 \\ (-)\ \log 1.204 = 0.0806 \\ \hline \log x = 0.1910; \end{array} \qquad \text{hence } \; x = 1.552.$$

★178. Logarithms to various bases

The base 10 is convenient for a system of logarithms when they are being used to simplify computation. The only base other than 10 which is used appreciably is the irrational number $e = 2.71828\cdots$, which is fully as important a constant in mathematics as the familiar number π. Logarithms to the base e are called **natural,** or **Naperian,** logarithms. Natural logarithms have many advantages over common logarithms for advanced theoretical purposes.

Recall that the equations $N = b^x$ and $x = \log_b N$ are equivalent. Hence, if N and b are given, we can find $\log_b N$ by solving the exponential equation $N = b^x$ by use of *common* logarithms. In particular, the natural logarithm of N can be found by solving $N = e^x$ for x.

EXAMPLE 1. Find $\log_e 35$.

SOLUTION. Let $x = \log_e 35$; then, $35 = e^x$. On taking the common logarithm of both sides we obtain $x \log_{10} e = \log_{10} 35$.

$$x = \frac{\log_{10} 35}{\log_{10} e} = \frac{1.5441}{0.4343};\qquad \begin{array}{rl} \log 1.544 = & 10.1886 - 10 \\ (-)\ \log .4343 = & 9.6378 - 10 \\ \hline \log x = & 0.5508. \end{array}$$

$$x = 3.555 = \log_e 35.$$

THEOREM I. *If a and b are any two bases, then*

$$\log_a N = (\log_a b)(\log_b N). \tag{1}$$

Proof. Let $y = \log_b N$; then $$N = b^y. \tag{2}$$

Hence, $$\log_a N = \log_a b^y = y \log_a b = (\log_a b)(\log_b N).$$

The number $\log_a b$ is called the **modulus** of the system of base a with respect to the system of base b. Given a table of logarithms to the base b, we could form a table of logarithms to the base a by multiplying each entry of the given table by $\log_a b$.

★EXERCISE 94

Solve for x or for n, or compute the specified logarithm.

1. $12^x = 28$. **2.** $51^x = 569$. **3.** $5^{2x} = 28(2^x)$. **4.** $15^{3x} = 85(3^x)$.

5. $.67^x = 8$. **6.** $.093^x = 12$. **7.** $(1.03)^{-n} = .587$. **8.** $(1.04)^n = 1.562$.

9. $\dfrac{(1.05)^x - 1}{.05} = 6.3282$. **10.** $\log 5x^2 + \log \dfrac{4}{x} = 5.673$.

11. $\log_e 75$. **12.** $\log_e 1360$. **13.** $\log_4 10$. **14.** $\log_{15} 33$. **15.** $\log_{.8} 23.8$.

16. Find the natural logarithm of (*a*) 4368.1; (*b*) 4.3681. (Notice that the results do *not* differ by an integer, so that Theorem I of page 245 does *not* hold for natural logarithms.)

★179. Graphs of logarithmic and exponential functions

We recall that $y = \log_a x$ and $x = a^y$ are equivalent relations. We call $\log_a x$ a *logarithmic function* of x and a^y an *exponential function* of y.

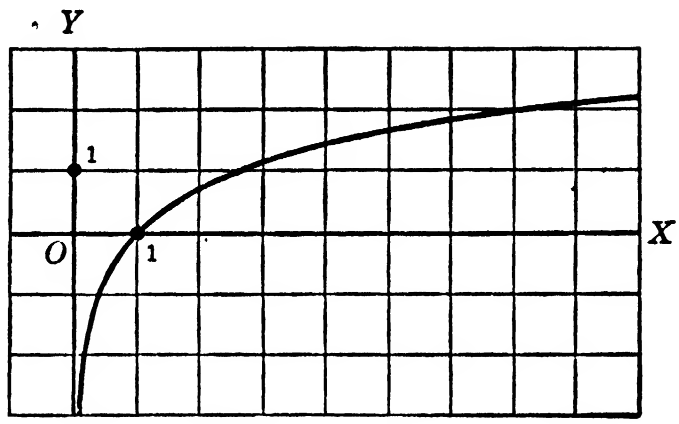

Fig. 18

ILLUSTRATION 1. In Figure 18, we have the graph of $y = \log_e x$. For any base $a > 1$, the graph of $\log_a x$ would be similar. This graph assists us in remembering the following facts.

I. *If x is negative, $\log_a x$ is not defined.*

II. *If $0 < x < 1$, $\log_a x$ is negative, and* $\log_a 1 = 0$.

III. *If x increases without limit, $\log_a x$ increases without limit; if x approaches zero, $\log_a x$ decreases without limit.*

Since $y = \log_a x$ is equivalent to $x = a^y$, *these equations have the same graph.* Thus, in Figure 18 we have a graph of $x = e^y$.

★EXERCISE 95

1. Graph $y = \log_{10} x$ for $0 < x \leqq 30$. From the graph, read the value of $10^{1.3}$; $10^{-.6}$; $10^{.3}$.

2. Graph $y = 2^x$ from $x = -6$ to $x = 4$. From the graph, read $\log_2 6$.

★180. Applications to compound interest

On page 235 we saw that, if $\$P$ is invested at the interest rate i compounded annually, the amount $\$A$ on hand at the end of n years is given by the formula

$$A = P(1 + i)^n.$$

ILLUSTRATION 1. If \$1000 is invested for 20 years at 5% compounded annually, the amount at the end of the time is

$$A = 1000(1.05)^{20} = 1000(2.653) = \$2653. \qquad \text{(From Table III)}$$

For unusual values of i, it is impossible to employ an interest table as in Illustration 1, but then logarithms can be used.

EXAMPLE 1. Compute $A = 2000(1.036)^{25}$.

SOLUTION. $\log 1.036 = 0.0153$; $\quad 25 \log 1.036 = 0.3825$

$$\begin{array}{rl} 25 \log 1.036 &= 0.3825 \\ \log 2000 &= 3.3010 \ (+) \\ \hline \log A &= 3.6835. \end{array}$$

Hence, $A = \$4826$.

We claim only three significant digits in the result because accuracy is lost in the multiplication 25 log 1.036. The 4th digit in 4826 is unreliable.

EXAMPLE 2. Solve for the rate i: $\quad 2500 = 2000(1 + i)^8$.

SOLUTION. 1. $(1 + i)^8 = \frac{2500}{2000}$, or $(1 + i)^8 = 1.250$.

2. Hence, $1 + i = \sqrt[8]{1.250}$; we compute this root by use of Table II.

$\log 1.250 = 0.0969$; $\quad \frac{1}{8} \log 1.250 = 0.0121$; $\quad$ hence, $\sqrt[8]{1.250} = 1.028$.

3. Therefore, $1 + i = 1.028$; $\quad$ or, $\quad i = 1.028 - 1 = .028 = 2.8\%$.

★EXERCISE 96

By use of Table III, find the compound amount at the end of the time if the money is invested at the specified rate compounded annually.

1. \$2500; at 4%, for 16 years. **2.** \$1200; at 6%, for 13 years.

3. \$1600; at 3%, for 35 years. **4.** \$400; at 5%, for 42 years.

From $A = P(1 + i)^n$, *we obtain* $P = A(1 + i)^{-n}$. *Use this result and Table* IV *to solve Problems* 5 *and* 6.

5. What principal should be invested now at 5% compounded annually to create \$2500 as the amount at the end of 15 years?

6. What principal should be invested now at 6% compounded annually to create \$1000 as the amount at the end of 26 years?

7. At what interest rate compounded annually will a \$2000 principal grow to the amount \$3500 at the end of 10 years?

8. At what interest rate compounded annually will a \$3000 principal double itself by the end of 15 years?

9. How long will it take \$300 to grow to the amount \$750 if invested at 5% compounded annually? (Recall Section 177.)

10. How long does it take money to double if invested at 6% compounded annually?

11. How long does it take money to double if invested at 6% simple interest? Compare with the result of Problem 10.

CHAPTER 16

SYSTEMS INVOLVING QUADRATICS

181. Graph of a quadratic equation in two variables

A **solution** of an equation in two variables x and y is a **pair of values** of the variables which satisfies the equation. The *graph* or *locus* of the equation is the set of all points whose coordinates, (x, y), form real-valued solutions of the equation.

EXAMPLE 1. Graph: $\qquad x^2 + y^2 = 25.$

SOLUTION. 1. When $x = 0$, $y^2 = 25$; $y = \pm 5$. Two solutions of the equation are $(0, 5)$ and $(0, -5)$.

2. When $y = 0$, $x^2 = 25$; $x = \pm 5$. Two solutions of the equation are $(5, 0)$ and $(-5, 0)$.

3. We plot the four points just found, with the same unit on OX and OY in Figure 19, and verify an advance inference that the graph is a *circle* whose center is the origin and radius is 5.

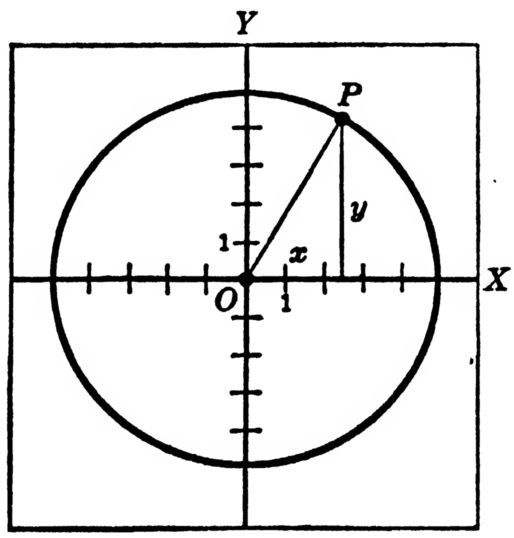

Fig. 19

Comment. Let P, with the coordinates (x, y), be any point in the coordinate plane with origin at O, in a system where the *same unit* is used in measuring *all* lengths. Then

$$x^2 + y^2 = (OP)^2.$$

Hence, if $x^2 + y^2 = 25$, the point P must lie on a circle about O as center with $(OP)^2 = 25$, or with radius 5.

EXAMPLE 2. Graph: $$9x^2 - 4y^2 = 36. \tag{1}$$

SOLUTION. 1. Solve for x in terms of y: $\qquad 9x^2 = 36 + 4y^2;$

$$x^2 = \frac{36 + 4y^2}{9}; \quad x = \pm\sqrt{\frac{36 + 4y^2}{9}};$$

$$x = \pm \tfrac{2}{3}\sqrt{9 + y^2}. \tag{2}$$

2. We assign values to y and compute the corresponding values of x. Thus, if $y = 0$, then $x = \pm \frac{2}{3}\sqrt{9} = \pm 2$. If $y = 3$, then

$$x = \pm \tfrac{2}{3}\sqrt{18} = \pm 2\sqrt{2} = \pm 2.8.$$

We tabulate the corresponding values of x and y in the following table.

(a)	$y =$	-6	-3	0	3	6
$x = \frac{2}{3}\sqrt{9 + y^2}$	$x =$	4.5	2.8	2	2.8	4.5
(b)	$y =$	-6	-3	0	3	6
$x = -\frac{2}{3}\sqrt{9 + y^2}$	$x =$	-4.5	-2.8	-2	-2.8	-4.5

We plot the points given by the pairs of values of x and y in the table. In Figure 20, the points listed for (a) in the table give the open curve FDE; the points for (b) give HBG. These two open curves, together, are called a **hyperbola,** and it is the graph of equation 1. Each piece of the hyperbola is called a *branch* of it.

Comment. The equation $9x^2 - 4y^2 = 36$ defines x as a *two*-valued function of y, as shown in (2), or y as a *two*-valued function of x. The graph of the equation consists of the graphs of the two *single*-valued irrational functions

$$x = +\tfrac{2}{3}\sqrt{9 + y^2} \quad \text{and} \quad x = -\tfrac{2}{3}\sqrt{9 + y^2}.$$

The graph of the first of these is the branch FDE and the graph of the second is HBG. The two branches *together* make up the graph of equation 1. The branches are *symmetrical* with respect to the y-axis.

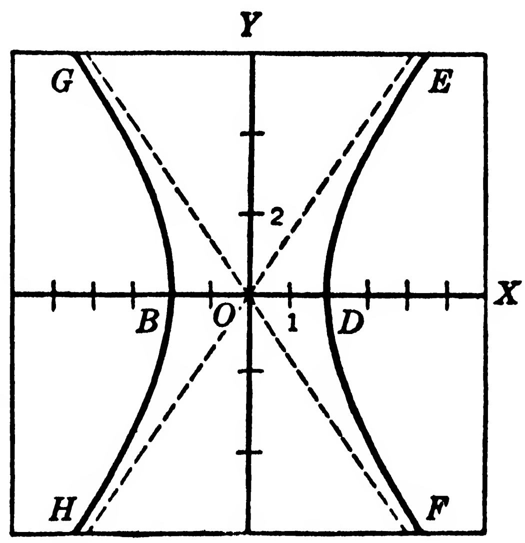

Fig. 20

Note 1. To every hyperbola there correspond two characteristic lines, called **asymptotes,** which are indicated by dotted lines in Figure 20. As we recede out on any branch of the hyperbola, the curve approaches the corresponding asymptote but *never reaches it.* By moving far enough out on the branch, we may approach the asymptote as closely as we please. It is proved in analytic geometry that the equations of the asymptotes for equation 1 are obtainable as follows:

1. *Replace the constant term in the equation by* 0, *and factor the left member:*

$$9x^2 - 4y^2 = 0; \quad (3x - 2y)(3x + 2y) = 0.$$

2. *Equate each factor separately to zero:*

$$3x - 2y = 0 \quad \text{and} \quad 3x + 2y = 0.$$

These are the equations of the asymptotes.

EXAMPLE 3. Graph: $$x^2 + 4y^2 = 25. \tag{3}$$

SOLUTION. 1. Solve for y:

$$y^2 = \tfrac{1}{4}(25 - x^2); \qquad y = \pm\tfrac{1}{2}\sqrt{25 - x^2}. \tag{4}$$

2. To obtain real values for y, the numerical value of x may not be allowed to exceed 5. Thus, if $x = \pm 8$, $\sqrt{25 - x^2} = \sqrt{-39}$, which is imaginary.

3. Place $x = 0$ in (3) to determine the y-intercepts:

$$4y^2 = 25; \qquad y^2 = \tfrac{25}{4}; \qquad y = \pm\tfrac{5}{2}.$$

Hence, two points on the graph are $(0, \tfrac{5}{2})$ and $(0, -\tfrac{5}{2})$, labeled A and C in Figure 21.

4. Place $y = 0$ in (3) to determine the x-intercepts:

$$x^2 = 25; \qquad x = \pm 5.$$

Hence, two points on the graph are (5, 0) and (− 5, 0), labeled B and D in Figure 21.

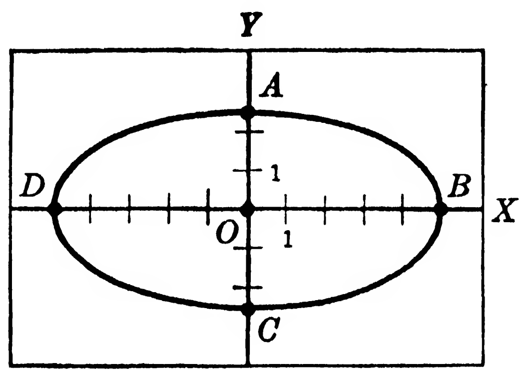

Fig. 21

5. As many more points as desired can be found by substituting values of x in (4) and computing the corresponding values of y. When the points are joined by a smooth curve we obtain the oval $ABCD$ in Figure 21. The curve is called an **ellipse.** The graph of the *positive* valued function $y = \tfrac{1}{2}\sqrt{25 - x^2}$, from (4), is the half of the ellipse *above* the x-axis. The graph of $y = -\tfrac{1}{2}\sqrt{25 - x^2}$ is the *lower* half, which is *symmetrical* to the upper half. The whole ellipse is the graph of (3).

The facts stated in the following summary are proved in more advanced mathematics.

SUMMARY. *The graph of any quadratic equation in two variables x and y with real solutions is either an ellipse, a hyperbola, a circle, a parabola, a pair of straight lines, or a single point.*

1. *If c is positive, the graph of $x^2 + y^2 = c$ is a circle whose radius is $\sqrt{c}$ and center is the origin, provided that the same unit is used on the scales of the x-axis and y-axis.*

2. *If a, b, and c have the same sign, the graph of $\mathbf{ax^2 + by^2 = c}$ is an ellipse, with center at the origin; if $a = b$, the ellipse is a circle, provided that the same unit is used on the scales of the x-axis and y-axis.*

3. *If a and b have opposite signs and if c is not zero, the graph of $\mathbf{ax^2 + by^2 = c}$ is a hyperbola.*

4. *If $c \neq 0$, the graph of $\mathbf{xy = c}$ is a hyperbola; if $c > 0$, one branch of the hyperbola lies wholly in quadrant* I, *and the other in quadrant* III; *if $c < 0$, the branches are in quadrants* II *and* IV, *respectively. The coordinate axes are the asymptotes of the hyperbola.*

5. *If a quadratic equation in x and y does not involve y^2 or xy, the graph of the equation is a parabola whose axis is parallel to the y-axis; if the equation does not involve x^2 or xy, the graph is a parabola whose axis is parallel to the x-axis.*

ILLUSTRATION 1. The graph of the equation $3x^2 + 7y^2 = 8$ is an *ellipse*, of $5x^2 - y^2 = 7$ is a *hyperbola*, and of $4x^2 + 4y^2 = 25$ is a *circle*, whose radius is $5/2$.

EXAMPLE 4. Determine the nature of the graph of

$$2x^2 - xy - 3y^2 = 0. \tag{5}$$

SOLUTION. 1. Factor: $\qquad (2x - 3y)(x + y) = 0.$

Hence, (5) is satisfied by values (x, y) in case

$$(a)\ 2x - 3y = 0, \qquad \text{or} \qquad (b)\ x + y = 0.$$

Therefore, the set of all points (x, y) satisfying (5) consists of those satisfying (a) and those satisfying (b). Or, in other words, the graph of (5) consists of the graph of (a) *and* the graph of (b).

2. The graphs of (a) and (b) are straight lines through the origin. Hence, the graph of (5) consists of these *two straight lines.*

Comment. Another case similar to Example 4 was met in finding the asymptotes in Example 2. They were the two straight lines which are the graph of the equation $9x^2 - 4y^2 = 0$.

EXAMPLE 5. Determine the nature of the graph of

$$4y - 3x^2 + 5x - 7 = 0.$$

SOLUTION. 1. Solve for y: $\qquad y = \frac{3}{4}x^2 - \frac{5}{4}x + \frac{7}{4}.$

2. Thus, y is a *quadratic function* of x, and therefore the graph of the given equation is a *parabola* whose axis is parallel to the y-axis. To graph the equation, we would compute the coordinates of the vertex of the parabola and proceed as in Section 132, page 186.

182. Routine for graphing

It is important to be able to construct reasonably good graphs *quickly*. Beyond this, it is also essential to have a procedure for improving on such graphs when the necessity arises. The following suggestions are of aid in constructing graphs quickly for equations of the second degree in x and y:

1. *Refer to the summary of Section* 181 *and if possible decide on the nature of the graph before carrying out details of the work.*

2. *When the graph of* $ax^2 + by^2 = c$ *is a circle, find its radius,* $\sqrt{\dfrac{c}{a}}$, *and construct the circle with compasses.*

3. *When the graph of* $ax^2 + by^2 = c$ *is an ellipse, find the x-intercepts by placing* $y = 0$ *and solving for x, and find the y-intercepts by placing* $x = 0$ *in the given equation. Then, sketch the ellipse through the four intercept points thus obtained.**

4. *When the graph of* $ax^2 + by^2 = c$ *is a hyperbola:*

   Find its asymptotes by replacing c by 0 *and constructing the two straight lines which are the graph of* $ax^2 + by^2 = 0$.

   Find the x-intercepts or the y-intercepts. (One set of intercepts will be imaginary because the hyperbola will cut just one of the coordinate axes.)

   Sketch the hyperbola through the real intercepts thus found, with each branch of the curve approaching the asymptotes smoothly.

5. *When a quadratic equation in* (x, y) *is linear in one variable, solve for it in terms of the other variable and then graph the resulting parabola by the method of Section* 132.

ILLUSTRATION 1. To graph $9x^2 - 4y^2 = 36$ *quickly*, we first note that the graph will be a *hyperbola*. We substitute $y = 0$ and find that the x-intercepts are *real*, $x = \pm 2$; we thus obtain points B and D in Figure 20, page 263. We obtain the asymptotes, as in Note 1, page 263. Then we sketch branches EDF and GBH through points B and D in Figure 20.

To improve on a graph as obtained through the preceding suggestions, or when doubt arises as to the nature of a graph, *solve the given equation for one variable in terms of the other and compute as many points as needed*, with Example 2 of Section 181 as a model.

* Illustrated in Example 3, page 264.

EXERCISE 97

Graph each equation on cross-section paper.

1. $x^2 + y^2 = 9$.
2. $9x^2 + 4y^2 = 36$.
3. $4x^2 - y^2 = 16$.
4. $4x^2 + 4y^2 = 9$.
5. $xy = 4$.
6. $xy = -6$.
7. $9x^2 - 16y^2 = 0$.
8. $x^2 + 4y^2 = 0$.
9. $4y^2 - 9x^2 = 36$.
10. $(6x - y)(3x + 2y) = 0$.
11. $y = x^2 - 4x + 7$.
12. $2y - 4x + 6x^2 = 9$.
13. $3x^2 + 4xy - 4y^2 = 0$.
14. $36 - x^2 - 9y^2 = 0$.
15. $(x - 2y)(3x - 2y - 6) = 0$.
16. Rewrite statements I and II of page 187 for the graph of

$$x = ay^2 + by + c,$$

where it is understood that the x-axis is to be *horizontal as usual.*

Graph each equation, with the aid of Problem 16.

17. $x = 4y^2$.
18. $x = 2y^2 + 8y - 6$.
19. $4y^2 + 2x - 24y + 9 = 0$.

183. Graphical solution of systems involving quadratics

EXAMPLE 1. Solve the following system graphically:

$$\begin{cases} x^2 - 2y^2 = 1, & (1) \\ x^2 + 4y^2 = 25. & (2) \end{cases}$$

SOLUTION. 1. We graph each equation, on one coordinate system. The graph of (1) is the hyperbola and the graph of (2) is the ellipse in Figure 22.

2. Any point on the hyperbola has coordinates which satisfy (1), and any point on the ellipse has coordinates which satisfy (2). Hence, both equations are satisfied by the coordinates of A, B, C, and D, which are the points of intersection of the ellipse and the hyperbola:

A: $(x = 3, y = 2)$.
B: $(x = -3, y = 2)$.
C: $(x = -3, y = -2)$.
D: $(x = 3, y = -2)$.

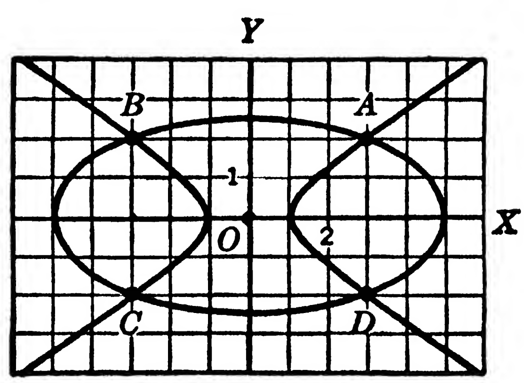

Fig. 22

These pairs of values are the solutions of the system [(1), (2)] and can be checked by substitution in the given equations.

Only real solutions can be found by the preceding graphical method and, usually, solutions can be read only approximately from a graph.

EXERCISE 98

Solve graphically.

1. $\begin{cases} x^2 + y^2 = 16, \\ x - 2y = 3. \end{cases}$
2. $\begin{cases} 2x + y = 3, \\ x^2 + y^2 = 9. \end{cases}$
3. $\begin{cases} x^2 - y^2 = 16, \\ x + y = 2. \end{cases}$
4. $\begin{cases} 4x^2 - y^2 = 16, \\ 3 = x - y. \end{cases}$
5. $\begin{cases} 4x^2 + 9y^2 = 36, \\ 4y^2 - x^2 = 4. \end{cases}$
6. $\begin{cases} 25x^2 + y^2 = 25, \\ 9y^2 - x^2 = 9. \end{cases}$
7. $\begin{cases} y = 2x^2 - 8x + 9, \\ xy = 12. \end{cases}$
8. $\begin{cases} 2x^2 - xy - 6y^2 = 0, \\ x^2 + y^2 = 4. \end{cases}$
9. $\begin{cases} x^2 + y^2 = 1, \\ 25x^2 + 4y^2 = 100. \end{cases}$
10. $\begin{cases} 9x^2 + 4y^2 = 36, \\ x^2 + y^2 = 9. \end{cases}$

184. Solution of a simple system

A system consisting of one linear and one quadratic equation in two unknowns x and y usually has either (a) two different real solutions, or (b) two real solutions which are the same, or (c) two imaginary solutions. These possibilities correspond, respectively, to the following geometrical possibilities: the straight line, which is the graph of the linear equation, (a) may cut the graph of the quadratic in two points, or (b) may be tangent to, or (c) may not touch the graph of the quadratic.

EXAMPLE 1. Solve:
$$\begin{cases} 4x^2 - 6xy + 9y^2 = 63, & (1) \\ 2x - 3y = -3. & (2) \end{cases}$$

SOLUTION. 1. Solve (2) for x:
$$x = \frac{3y - 3}{2}. \qquad (3)$$

2. Substitute (3) in (1):
$$4\left(\frac{3y - 3}{2}\right)^2 - 6y\left(\frac{3y - 3}{2}\right) + 9y^2 = 63. \qquad (4)$$

$$y^2 - y - 6 = 0; \quad (y - 3)(y + 2) = 0; \quad y = 3 \quad \text{and} \quad y = -2.$$

3. In (3), if $y = 3$, then $x = 3$; if $y = -2$, then $x = -9/2$.

4. The solutions are $\boxed{x = 3, y = 3}$ and $\boxed{x = -\frac{9}{2}, y = -2}$.

Note 1. Since a solution of a *pair* of equations in x and y is a *pair* of related values of x and y, it is very essential that each solution should be plainly indicated as a *pair of values*, as in Example 1.

Note 2. A system of the type considered in this section will hereafter be called a **simple system**.

SUMMARY. *To solve a system of one linear and one quadratic equation in x and y algebraically:*

1. *Solve the linear equation for one unknown in terms of the other, say for y in terms of x, and substitute the result in the quadratic equation; this eliminates one unknown.*

2. *Solve the quadratic equation obtained in Step 1 and, for each value of the unknown obtained, find the corresponding value of the other unknown by substitution in the given linear equation.*

EXERCISE 99

Solve, (a) graphically and (b) algebraically.

1. $\begin{cases} x^2 + y^2 = 25, \\ y + x = -1. \end{cases}$
2. $\begin{cases} x^2 + y^2 = 169, \\ x + y = 7. \end{cases}$
3. $\begin{cases} x^2 - y^2 = 16, \\ 3y + 5x = 16. \end{cases}$
4. $\begin{cases} c^2 + d^2 = 100, \\ 4d + 3c = 50. \end{cases}$
5. $\begin{cases} u^2 + 2v^2 = 8, \\ u + 2v = 6. \end{cases}$
6. $\begin{cases} 3a^2 + 4b^2 = 9, \\ a + 2b = 4. \end{cases}$

Solve algebraically.

7. $\begin{cases} 2x + y + 3 = 0, \\ 2x^2 + y^2 - 6y = 9. \end{cases}$
8. $\begin{cases} x^2 + 9y^2 = 25, \\ x - 7 + 3y = 0. \end{cases}$
9. $\begin{cases} 5a - 4b = 12, \\ a^2 + 2a - b^2 - 4b = 12. \end{cases}$
10. $\begin{cases} 3x + 8y = 26, \\ x^2 + 2x - 4y = 23 - 4y^2. \end{cases}$
11. $\begin{cases} 5xy = 2x + y, \\ 4x + 2y = 5. \end{cases}$
12. $\begin{cases} 2xy + 3y + 4x - 1 = 0, \\ 2x + y + 3 = 0. \end{cases}$
13. $\begin{cases} y - 2x + 1 = 0, \\ x^2 - 4x - 4y^2 + 24y = 36. \end{cases}$
14. $\begin{cases} 2x + y = 2a - 1, \\ 4x^2 + 4x + y^2 = 2a^2 + 1. \end{cases}$
15. $\begin{cases} 2x + y = 2, \\ 4x^2 - 2xy + y^2 + 8x - 2y = 3. \end{cases}$
16. $\begin{cases} 2x + y = 2a + b, \\ 4x^2 + y^2 - 2by = 2a^2 + b^2. \end{cases}$

185. Solution of a system of two quadratic equations

When both equations of a system are quadratic, the system usually has *four different solutions,* all or two of which may involve imaginary numbers. The student should recall his graphical solutions of systems of this type where four solutions were obtained.

Note 1. The fact stated in the preceding paragraph is a special case of the following theorem which is proved in a later course in algebra:

A system of two integral rational equations in x and y, in which one equation is of degree m and the other is of degree n in x and y, *usually has mn solutions.*

Thus, a system consisting of an equation of the third degree and a quadratic usually has 3×2 or 6 solutions. Usually, the algebraic solution of two simultaneous quadratics brings in the solution of a fourth degree equation in one variable. At this stage in algebra, the student is able to solve only very simple fourth degree equations. Hence he is not prepared to consider the solution of *all* systems of simultaneous quadratics. Therefore, in this chapter we consider only special elementary types of systems.

186. Systems linear in the squares of the variables

When both equations have the form $ax^2 + by^2 = c$, the system is linear in x^2 and y^2 and can be solved for x^2 and y^2 by the methods applicable to systems of linear equations.

EXAMPLE 1. Solve: $\begin{cases} x^2 + y^2 = 25, & (1) \\ x^2 + 2y^2 = 34. & (2) \end{cases}$

SOLUTION. 1. Multiply by 2 in (1): $2x^2 + 2y^2 = 50.$ (3)

2. Subtract, (3) − (2): $x^2 = 16;\ x = \pm 4.$

3. Substitute $x^2 = 16$ in (1): $16 + y^2 = 25;\quad y^2 = 9;\quad y = \pm 3.$

4. Hence, if x is either $+4$ or -4, we obtain as corresponding values $y = +3$ and $y = -3$, and there are *four* solutions of the system.

$\boxed{x = 4, y = 3}$; $\boxed{x = -4, y = 3}$; $\boxed{x = 4, y = -3}$; $\boxed{x = -4, y = -3}$.

EXERCISE 100

Solve each system, (a) graphically and (b) algebraically.

1. $\begin{cases} x^2 + y^2 = 4, \\ x^2 + 9y^2 = 9. \end{cases}$
2. $\begin{cases} x^2 - 9y^2 = 36, \\ x^2 + y^2 = 36. \end{cases}$
3. $\begin{cases} 4x^2 - y^2 = 16, \\ 9x^2 + 9y^2 = 16. \end{cases}$

Solve algebraically.

4. $\begin{cases} x^2 + 4y^2 = 14, \\ x^2 = 8y^2 - 16. \end{cases}$
5. $\begin{cases} x^2 - y^2 = 4, \\ 2x^2 + y^2 = 11. \end{cases}$
6. $\begin{cases} 2x^2 - 3y^2 = 3, \\ 5x^2 + 2y^2 = 17. \end{cases}$
7. $\begin{cases} 9x^2 = 8y^2 - 6, \\ 8x^2 - 3y^2 = 7. \end{cases}$
8. $\begin{cases} 15c^2 = 8 + 4d^2, \\ 15 - 12d^2 = 20c^2. \end{cases}$
9. $\begin{cases} 2t^2 = 6r^2 - 3, \\ 6 = 3t^2 + 5r^2. \end{cases}$
10. $\begin{cases} 7r^2 + 8s^2 = 36, \\ 11r^2 + 5s^2 = -4. \end{cases}$
11. $\begin{cases} 7x^2 - 6y^2 = 63, \\ 9x^2 + 2y^2 = 13. \end{cases}$
12. $\begin{cases} 6x^2 + 16 = 9y^2, \\ 4x^2 + 9y^2 = -4. \end{cases}$

187. Reduction to simpler systems

EXAMPLE 1. Solve:
$$\begin{cases} x^2 + y^2 = 14, & (1)\\ x^2 - 3xy + 2y^2 = 0. & (2)\end{cases}$$

SOLUTION. 1. Factor (2): $$(x - 2y)(x - y) = 0. \tag{3}$$

2. Therefore, (2) is satisfied if either $x - 2y = 0$, or $x - y = 0$.

3. Hence, (1) and (2) are satisfied if and only if x and y satisfy one of the following systems:

$$\text{I.}\ \begin{cases} x^2 + y^2 = 14,\\ x - y = 0.\end{cases} \qquad \text{II.}\ \begin{cases} x^2 + y^2 = 14,\\ x - 2y = 0.\end{cases}$$

4. On solving (I) by the method of Section 184, we obtain two solutions: $(x = \sqrt{7}, y = \sqrt{7})$ and $(x = -\sqrt{7}, y = -\sqrt{7})$. From (II) we obtain

$$(x = \tfrac{2}{5}\sqrt{70}, y = \tfrac{1}{5}\sqrt{70}); \qquad (x = -\tfrac{2}{5}\sqrt{70}, y = -\tfrac{1}{5}\sqrt{70}).$$

We say that the given system in Example 1 is *equivalent* to the systems I and II because the solutions of the given system consist of the solutions of (I) together with those of (II).

The preceding method applies if, after writing each equation with one member *zero*, we can factor at least one of the other members.

188. Elimination of constants

A system in which *all terms involving the variables are of the second degree* can sometimes be solved by use of the equation we obtain on eliminating the constant terms from the original system.

EXAMPLE 1. Solve:
$$\begin{cases} x^2 + 3xy = 28, & (1)\\ xy + 4y^2 = 8. & (2)\end{cases}$$

INCOMPLETE SOLUTION. 1. *Eliminate the constants.*

Multiply (1) by 2: $$2x^2 + 6xy = 56. \tag{3}$$

Multiply (2) by 7: $$7xy + 28y^2 = 56. \tag{4}$$

Subtract, (3) − (4): $$2x^2 - xy - 28y^2 = 0;\ \text{or,}$$
$$(2x + 7y)(x - 4y) = 0. \tag{5}$$

2. To solve [(1), (2)] we may now solve [(2), (5)]. This system is equivalent to the following simpler systems:

$$\begin{cases} xy + 4y^2 = 8,\\ 2x + 7y = 0.\end{cases} \qquad \begin{cases} xy + 4y^2 = 8,\\ x - 4y = 0.\end{cases} \tag{6}$$

Instead of using (2) in (6) we could equally well have used (1). Each system in (6) has *two* solutions and thus [(1), (2)] has *four* solutions.

EXERCISE 101

Solve algebraically and graphically.

1. $\begin{cases} x^2 + y^2 = 4, \\ (x + y)(x - 2y) = 0. \end{cases}$

2. $\begin{cases} x^2 + y^2 = 9, \\ (x - y)(x + 3y) = 0. \end{cases}$

Note 1. Hereafter in this chapter, leave all surd values in radical form. Moreover, unless otherwise stated, *to solve* a system will mean to solve *algebraically*.

Solve by reducing to simpler systems.

3. $\begin{cases} 2x^2 + 5xy - 3y^2 = 0, \\ 2x^2 + 3xy = 2. \end{cases}$

4. $\begin{cases} 3x^2 + 5xy - 2y^2 = 0, \\ x^2 + xy = 4. \end{cases}$

5. $\begin{cases} 2x^2 + 7xy + 6y^2 = 0, \\ x^2 + 3y^2 = 7. \end{cases}$

6. $\begin{cases} 3xy + 2y^2 = 0, \\ 2x^2 - xy - 2y^2 = 8. \end{cases}$

Solve by eliminating the constant terms.

7. $\begin{cases} x^2 + 3xy = 28, \\ xy + 4y^2 = 8. \end{cases}$

8. $\begin{cases} x^2 - 5xy + 6y^2 = 10, \\ x^2 - xy = 4. \end{cases}$

9. $\begin{cases} 6cd + 2d^2 = -7, \\ 2c^2 - 2cd = 15. \end{cases}$

10. $\begin{cases} x^2 - xy = 1, \\ 2x^2 + 2y^2 = 5. \end{cases}$

11. $\begin{cases} 2u^2 + uz + 5z^2 = 11, \\ uz - z^2 + 3 = 0. \end{cases}$

12. $\begin{cases} x^2 + 3y^2 = 28, \\ x^2 - xy + 4y^2 = 40. \end{cases}$

13. $\begin{cases} x^2 - xy + 6 = 0, \\ xy + y^2 = 35. \end{cases}$

14. $\begin{cases} 2n^2 + 3mn = 1, \\ 9m^2 + 8n^2 = 9. \end{cases}$

15. $\begin{cases} m^2 + 2mn = 84, \\ 2mn + n^2 = 64. \end{cases}$

16. $\begin{cases} x^2 - 5xy - y^2 + 7 = 0, \\ x^2 - 3xy - 2y^2 + 4 = 0. \end{cases}$

★189. Additional devices for reducing to simpler systems

EXAMPLE 1. Solve:
$$\begin{cases} x^3 + y^3 = 27, & (1) \\ x + y = 3. & (2) \end{cases}$$

SOLUTION. 1. Factor (1):
$$(x + y)(x^2 - xy + y^2) = 27. \qquad (3)$$

2. Divide, (3) by (2): $\qquad x^2 - xy + y^2 = 9. \qquad (4)$

3. Hence, (x, y) satisfies [(1), (2)] if and only if (x, y) satisfies
$$\begin{cases} x + y = 3, & (5) \\ x^2 - xy + y^2 = 9. & (6) \end{cases}$$

The student should complete the solution by solving [(5), (6)] by the method of Section 184.

EXAMPLE 2. Solve:
$$\begin{cases} x^2 + xy + y^2 = 20, & (7) \\ xy = 5. & (8) \end{cases}$$

INCOMPLETE SOLUTION. 1. Add, (7) + (8): $x^2 + 2xy + y^2 = 25.$ (9)

2. From (9), $(x + y)^2 = 25$; hence $x + y = 5$, or $x + y = -5$.

3. To solve [(7), (8)], we would solve each of the following systems:
$$\begin{cases} x + y = 5, \\ xy = 5. \end{cases} \qquad \begin{cases} x + y = -5, \\ xy = 5. \end{cases}$$

★190. Determination of tangents to curves

EXAMPLE 1. Find the value of the constant k so that the graphs of the equations in the following system will be tangent:
$$\begin{cases} x^2 + y^2 = k^2, & (1) \\ x + y = 1. & (2) \end{cases}$$

SOLUTION. 1. If the graph of (2) is tangent to the graph of (1), then the two solutions of the system [(1), (2)] must be identical.

2. Substitute $y = 1 - x$ in (1): $2x^2 - 2x + (1 - k^2) = 0.$ (3)

3. From Step 1, we notice that (3) must have *equal roots.* Hence, *its discriminant must be zero*, or
$$(-2)^2 - 4(2)(1 - k^2) = 0; \quad 8k^2 - 4 = 0; \quad k = \pm \tfrac{1}{2}\sqrt{2}.$$
Thus, the straight line is tangent to the circle if its radius is .707.

★191. Equations symmetrical in x and y.

An equation in x and y is said to be *symmetrical* in x and y in case the equation is unaltered when x and y are *interchanged.* A quadratic equation in x and y is symmetrical in x and y *if the coefficients of x^2 and y^2 are equal and those of x and y are equal.* The method of the next example applies where each equation is symmetrical in the unknowns.

EXAMPLE 1. Solve:
$$\begin{cases} x^2 + y^2 + 2x + 2y = 8, & (1) \\ 2xy + x + y = -4. & (2) \end{cases}$$

INCOMPLETE SOLUTION. 1. Substitute $x = u + v;\ y = u - v.$ (3)

From (1): $2u^2 + 2v^2 + 4u = 8,$ (4)

From (2): $2u^2 - 2v^2 + 2u = -4.$ (5)

2. Solve the system [(4), (5)] for u and v:

Eliminate v^2, [(4) + (5)]: $4u^2 + 6u - 4 = 0.$ (6)

3. Solve (6) for u; then obtain v from (4). Each pair of values (u, v) when placed in (3) gives a solution of [(1), (2)].

★EXERCISE 102

Solve by any convenient method.

1. $\begin{cases} 2a + b = 2, \\ 8a^3 + b^3 = 98. \end{cases}$
2. $\begin{cases} a^3 - b^3 = 27, \\ a - b = 3. \end{cases}$
3. $\begin{cases} x(x + y) = 40, \\ y(x + y) = 20. \end{cases}$
4. $\begin{cases} x^2 + xy + y^2 = 7, \\ x^3 - y^3 = 35. \end{cases}$
5. $\begin{cases} x^2y + 2xy^2 = -24, \\ x + 2y - 4 = 0. \end{cases}$
6. $\begin{cases} x^2 + 2xy + y^2 = 4, \\ xy + 3x + 6 = 0. \end{cases}$
7. $\begin{cases} u^2 - 4uv + 4v^2 = 9, \\ 3u^2 + uv + v = 3. \end{cases}$
8. $\begin{cases} x^2 + y^2 = 13, \\ xy = 6. \end{cases}$
9. $\begin{cases} 4x^2 + 3xy + y^2 = 8, \\ xy = 1. \end{cases}$
10. $\begin{cases} 4x^2 - 2xy + 3y = 2, \\ 3y - 2xy + y^2 = 2. \end{cases}$
11. $\begin{cases} (x + y)^2 + x + y = 12, \\ x^2 + y^2 = 5. \end{cases}$
12. $\begin{cases} x^2 + xy - 3x = 8, \\ 3y - xy - y^2 = 4. \end{cases}$
13. $\begin{cases} x^2 + xy + y^2 = 61, \\ xy = 29 - x - y. \end{cases}$
14. $\begin{cases} x + 2y + 2z = 3, \\ 2z - 2x - y = 6, \\ x^2 + y^2 + z = 14. \end{cases}$
15. $\begin{cases} 4x^2 - y^2 - z^2 = 4, \\ 4x^2 - 2y^2 = 1, \\ 4x^2 + y^2 - 5z^2 = 8. \end{cases}$

Find the values of k for which the graphs are tangent. Then, if k is real, graph the equations of the resulting system.

16. $\begin{cases} x^2 + y^2 = 16, \\ kx - y = 5. \end{cases}$
17. $\begin{cases} x^2 + 4y^2 = 25, \\ 8y + 3x = k. \end{cases}$
18. $\begin{cases} x^2 + 4y^2 = 25, \\ 4 + 2y + kx = 0. \end{cases}$

Find an expression for c in terms of the other constants in case the graphs of the two equations in the variables (x, y) are tangent.

19. $\begin{cases} y = mx + c, \\ 9x^2 + 4y^2 = 36. \end{cases}$
20. $\begin{cases} y = mx + c, \\ a^2x^2 - b^2y^2 = a^2b^2. \end{cases}$
21. $\begin{cases} x = my + c, \\ a^2y^2 + b^2x^2 = a^2b^2. \end{cases}$

Solve by the method applying to symmetrical equations.

22. $\begin{cases} x^2 - 4x + y^2 - 4y = 17, \\ xy + 6 = 0. \end{cases}$
23. $\begin{cases} x^2 - 3xy + y^2 = 1, \\ 2x^2 - xy + 2y^2 = 17. \end{cases}$
24. $\begin{cases} x^2 + y^2 + xy + 2x + 2y = 5, \\ 5x^2 + 5y^2 + 2xy + 8x + 8y = 24. \end{cases}$

Solve without first clearing of fractions.

25. $\begin{cases} \dfrac{1}{x} - \dfrac{1}{y} = 11, \\ \dfrac{1}{xy} + 28 = 0. \end{cases}$
26. $\begin{cases} \dfrac{1}{x^2} + \dfrac{3}{y^2} = 7, \\ \dfrac{3}{x^2} - \dfrac{2}{y^2} = 10. \end{cases}$
27. $\begin{cases} \dfrac{1}{x^2} - \dfrac{3}{y^2} = 1, \\ \dfrac{2}{x^2} - \dfrac{1}{y^2} = 7. \end{cases}$

MISCELLANEOUS EXERCISE 103

Solve graphically.

1. $\begin{cases} 4x^2 + y^2 = 25, \\ 2x + y = 7. \end{cases}$
2. $\begin{cases} x^2 + 4y^2 = 16, \\ x^2 - y^2 = 9. \end{cases}$
3. $\begin{cases} x^2 + y^2 = 20, \\ x^2 - 4y^2 = 4. \end{cases}$
4. $\begin{cases} (x - y - 2)(x - y - 1) = 0, \\ x^2 + y^2 = 10. \end{cases}$
5. $\begin{cases} y = 1 - x^2, \\ 4x = y^2 + 2y - 7. \end{cases}$

6–8. Solve Problems 1, 2, and 4 algebraically and compare the results with the solutions previously obtained.

Graph each equation.

9. $4x^2 + 9y^2 = 0.$ **10.** $4x^2 - 9y^2 = 0.$ **11.** $4x^2 - 9y^2 = 36.$

Solve algebraically.

12. $\begin{cases} x^2 + y^2 + 6y - 2x + 1 = 0, \\ y - 2x + 2 = 0. \end{cases}$
13. $\begin{cases} 3c + 2d = -2, \\ cd + 8c = 4. \end{cases}$
14. $\begin{cases} (2x + y)(x - y - 1) = 0, \\ 4x^2 + 4xy + 8x - 4y = 3y^2. \end{cases}$
15. $\begin{cases} 3xy + y^2 = 0, \\ 4x^2 - xy - y^2 = 4. \end{cases}$
16. $\begin{cases} 2y^2 - xy = 16, \\ x^2 - xy - y^2 = 20. \end{cases}$
17. $\begin{cases} 3u^2 + uv - v^2 = 9, \\ v^2 - u^2 = 27. \end{cases}$
18. $\begin{cases} x^2 - 3xy + 2y^2 = 3, \\ x^2 - y^2 + 3xy = 3. \end{cases}$
19. $\begin{cases} 2r^2 - 5rh - 3h^2 = 0, \\ r^2 - rh = 75. \end{cases}$

Solve for x and y.

20. $\begin{cases} 4x^2 + 4y^2 = 2a^2 + 2ab + b^2, \\ 8x^2 - 4y^2 = a^2 - 2ab - b^2. \end{cases}$
21. $\begin{cases} x + y = a, \\ 2x^2 + 2y^2 = a^2 + 1. \end{cases}$

Solve each problem by introducing two unknowns.

22. The sum of two numbers is 28 and the sum of their squares is 634. Find the numbers.

23. Find the dimensions of a rectangle whose area is 60 square feet and whose diagonal is 13 feet long.

24. The area of a rectangle is 55 square feet and its perimeter is 49 feet. Find the lengths of the sides of the rectangle.

25. The sum of the reciprocals of two numbers is 3 and the product of the numbers is $\frac{2}{14}$. Find the numbers.

26. A man divides $500 between two investments at simple interest, a first part at twice the interest rate obtained on the second part. The first investment grows to the amount $345 in 3 years, and the second to the amount $220 in 4 years. Find the interest rates and the sums invested.

27. The sum of the squares of the two digits of a positive integral number is 65 and the number is 9 times the sum of its digits. Find the original number.

28. Some men row 15 miles downstream on a river to a mountain and then climb 12 miles to its summit. They take 9 hours for the journey and, the next day, 9 hours to return. Find the rate at which they row in still water and their speed in ascending the mountain, if they descended 1 mile per hour faster than in ascending, and if the rate of the current of the river is 1 mile per hour.

29. A weight on one side of a lever balances a weight of 6 pounds placed 4 feet from the fulcrum on the other side. If the unknown weight is moved 2 feet nearer the fulcrum, the weight balances 2 pounds placed 9 feet from the fulcrum on the other side. Find the unknown weight.

30. A farmer has 3 fields of equal size and pays each of his workmen $4 per day. He paid a total of $42 to 2 workmen after each, working alone, plowed one of the fields. These men took $2\frac{4}{7}$ days to plow the third field when working together. How many days did it take each man to plow a field alone?

31. Two towns on opposite sides of a lake are 33 miles apart by water. At 6 A.M., from each town a boat starts for the other town, traveling at uniform speed. The boats pass each other at 9 A.M. One boat arrives at its destination 1 hour and 6 minutes earlier than the other. Find the time it takes each boat to make the trip across.

32. A wheel of one automobile makes 96 more revolutions per mile than a wheel of a second automobile. If 20 inches were added to the length of the radius of a wheel of the first automobile, the result would be the diameter of a wheel of the second automobile. Find the diameter of a wheel of each automobile, using 22/7 as the approximate value of π.

APPENDIX

NOTE 1. THE IRRATIONALITY OF $\sqrt{2}$

If there exists a rational number which is a square root of 2, then there exist two positive integers m and n, such that

$$\sqrt{2} = \frac{m}{n}, \tag{1}$$

where $\frac{m}{n}$ is a fraction in lowest terms. In other words, if $\sqrt{2}$ is rational there exist two integers m and n, **without a common factor,** such that (1) is true. Let us show that this assumption leads to a contradiction.

1. Square both sides of (1):

$$2 = \frac{m^2}{n^2}; \quad \text{or,}$$

$$2n^2 = m^2. \tag{2}$$

We see that 2 is a factor of the left member of $2n^2 = m^2$; hence 2 is a factor of the right member. Therefore 2 is a factor of m because otherwise 2 could not be a factor of m^2. That is, $m = 2k$, where k is some positive integer.

2. Place $m = 2k$ in (2):

$$2n^2 = (2k)^2 = 4k^2;$$

$$n^2 = 2k^2. \tag{3}$$

Consider $n^2 = 2k^2$; since 2 is a factor of the right member, hence 2 is a factor of n.

3. We have shown in Steps 1 and 2 that m and n have 2 as a factor. This contradicts our original assumption that m and n had no common factor. Hence, the assumed equation 1 has led us to a contradiction, and it follows that (1) itself must be false. Therefore no rational number exists which is a square root of 2, or, *$\sqrt{2}$ is an irrational number.*

Comment. We easily verify that $(1.4)^2 = 1.96$; $(1.41)^2 = 1.9881$; $(1.414)^2 = 1.999396$; etc. On considering the sequence of numbers

$$1.4, \quad 1.41, \quad 1.414, \quad 1.4142, \quad 1.41421, \cdots, \tag{4}$$

we see that the square of each number in (4) is less than 2 but that, on proceeding to the right in (4), the squares of the numbers approach 2 as a limit. Each number in (4) is a rational number; we refer to these numbers in (4) as the successive decimal approximations to $\sqrt{2}$.

NOTE 2. EXTENSION OF THE INDEX LAWS TO RATIONAL EXPONENTS

A complete proof that the index laws hold for any rational exponents could be constructed by showing, in succession, that the laws hold if the exponents are (1) any positive rational numbers and (2) zero, or positive or negative rational numbers. Without giving a complete discussion, we shall indicate the nature of the methods involved by proving some of the necessary theorems. For convenience in details, we shall assume that the *base* is positive. In our proofs, we use the index laws for positive integral exponents and the definitions of Sections 113, 114, and 115.

THEOREM I. *If m, n, and p are positive integers,* $\left(a^{\frac{m}{n}}\right)^p = a^{\frac{mp}{n}}$.

Proof. $\left(a^{\frac{m}{n}}\right)^p = \left[\left(a^{\frac{1}{n}}\right)^m\right]^p = \left(a^{\frac{1}{n}}\right)^{mp}$; [(3), page 151; (II), page 142]

or, $\left(a^{\frac{m}{n}}\right)^p = a^{\frac{mp}{n}}$. [(3), page 151]

THEOREM II. *If m, n, p, and q are positive integers, then*

$$a^{\frac{m}{n}}a^{\frac{p}{q}} = a^{\frac{m}{n}+\frac{p}{q}} = a^{\frac{mq+np}{nq}}.$$

Proof. $\left(a^{\frac{m}{n}}a^{\frac{p}{q}}\right)^{nq} = \left(a^{\frac{m}{n}}\right)^{nq}\left(a^{\frac{p}{q}}\right)^{nq}$ [(IV), page 143]

$= a^{mq}a^{pn}$. (Theorem I)

Hence, $\left(a^{\frac{m}{n}}a^{\frac{p}{q}}\right)^{nq} = a^{mq+pn}$. [(I), page 142]

Therefore, by the definition of an nqth root,

$$a^{\frac{m}{n}}a^{\frac{p}{q}} = \left(a^{mq+pn}\right)^{\frac{1}{nq}} = a^{\frac{mq+pn}{nq}}.$$

THEOREM III. $\left(a^{\frac{m}{n}}\right)^{\frac{p}{q}} = a^{\frac{mp}{nq}}$.

Suggestion for proof. Compute $\left[\left(a^{\frac{m}{n}}\right)^{\frac{p}{q}}\right]^{nq}$.

In the remainder of this note we shall assume that the index laws have been completely established for all positive rational exponents.

THEOREM IV. *Law* I *of Section* 105 *holds if the exponents are any positive or negative rational numbers.*

Comment. We are assuming that Law I has been established if both exponents are positive. Hence, it remains to show that, if h and k are any positive rational numbers, then $a^{-h}a^{-k} = a^{-h-k}$, and $a^h a^{-k} = a^{h-k}$.

Incomplete proof. By the definition of a negative power,

$$a^{-h}a^{-k} = \frac{1}{a^h}\cdot\frac{1}{a^k} = \frac{1}{a^{h+k}}; \quad \text{or,} \quad a^{-h}a^{-k} = a^{-(h+k)}.$$

NOTE 3. ABRIDGED MULTIPLICATION

The following example illustrates a method for abbreviating multiplication of numbers with many significant digits when the result is desired with accuracy only to a specified number of places.

Example 1. Compute 11.132157(893.214), accurate to two decimal places.

Solution. 1. To multiply by 893.214, multiply in succession by 800, 90, 3, .2, .01, and .004 and add the results (in ordinary multiplication these operations are in *reverse order*). Since we desire accuracy in the *second decimal place*, we carry *two extra places*, or *four decimal places*, in all items.

2. In the abridged method, to multiply by 800 we multiply by 8 and move the decimal point; all digits of 11.132157 are used in order to obtain four significant decimal places. *This first operation accurately locates the decimal point for the rest of the items.*

Ordinary Method	Abridged Method	*Multiply by*
11.132157	√√√√√ 11.132157	
893.214	893.214	
44528628	8905.7256	800
11132157	1001.8935	90
2 2264314	33.3963	3
33 396471	2.2264	.2
1001 89413	.1113	.01
8905 7256	.0444	.004
9943.398482598	9943.3975	*Add*
Result = 9943.40	*Result* = 9943.40	

3. To obtain *four* decimal places when multiplying by 90 we *do not need the last digit of* 11.132157; to indicate this place "√" over "7" and multiply 11.13215 by 90. Next, place "√" over "5" and multiply 11.1321 by 3; then "√" over "1" at the right and multiply 11.132 by .2; then "√" over "2" and multiply 11.13 by .01; then "√" over "3" and multiply 11.1 by .004. Then add and round off to two decimal places, obtaining 9943.40.

Note 1. The advantages of the preceding abridged method are obvious. As compared with the ordinary method, less labor is involved, the decimal point is accurately located, and fewer mistakes will occur in the final addition.

Note 2. An abridged method of division can be developed similar in principle to the method of abridged multiplication above.

NOTE 4. A FALLACIOUS PROOF THAT 2 = 1

The following absurd result that $2 = 1$ illustrates the contradictions that arise if division by zero occurs.

1. Suppose that	$y = b.$
2. Multiply by y:	$y^2 = by.$
3. Subtract b^2:	$y^2 - b^2 = by - b^2.$
4. Factor:	$(y - b)(y + b) = b(y - b).$
5. Divide by $(y - b)$:	$y + b = b.$
6. Since $y = b$ (Step 1),	$b + b = b$, or $2b = b.$
7. On dividing both sides by b, we obtain	$2 = 1.$

Discussion. In Step 5 we divided by zero, because $y - b = 0$ if $y = b$. Hence, Steps 5, 6, and 7 are not valid, because division by zero is not allowed.

NOTE 5. THE SQUARE ROOT PROCESS OF ARITHMETIC

Consider finding $\sqrt{7569}$. Since the radicand has *four* digits to the left of the decimal point, the square root must have *two* digits to the left of the decimal point, because the square of any number of *units* is *less than* 100, and the square of any number of *hundreds* is *greater than* 10,000. We observe that 80 is the largest whole number of tens whose square is *less than* 7569. Hence, we consider finding x, a number of units, so that

$$(80 + x)^2 = 7569. \tag{1}$$

By the formula for $(a + b)^2$, from (1) we obtain

$$6400 + 2\cdot 80\cdot x + x^2 = 7569; \tag{2}$$

$$160x + x^2 = 7569 - 6400;$$

$$160x + x^2 = 1169. \tag{3}$$

From (3),

$$x(160 + x) = 1169,$$

or

$$x = \frac{1169}{160 + x}. \tag{4}$$

An approximation to (4) is obtained if we use the **trial divisor** 160 in place of $(160 + x)$; this gives

$$x = \frac{1169}{160} = 7^+. \tag{5}$$

Then, we take $x = 7$ and verify that the **complete divisor,** $(160 + x)$ or 167, gives

$$\frac{1169}{167} = 7, \textit{ exactly}.$$

Hence, $\sqrt{7569} = 80 + 7 = 87$. We verify by squaring that $87^2 = 7569$.

In the following examples, the student will observe, in brief form, steps corresponding to those just explained in detail by reference to the formula for $(a + b)^2$. Hereafter, the details will be carried out without making the possible contacts with the square of a binomial. Essentially, at each stage of the following arithmetical process, we have knowledge of a in a binomial $(a + b)$ and we obtain *an approximation to b* so that $(a + b)^2$ will be *as nearly equal as convenient*, at that stage, *to the number whose square root is being obtained.*

EXAMPLE 1. Find $\sqrt{7569}$.

SOLUTION. Arrange 7569 into groups of two figures each, starting at the decimal point. After each of the following steps, read the corresponding explanation below.

Step 1
$$\begin{array}{r|l} & 8 \\ \hline & 75\ 69 \\ & 64 \\ \hline \end{array}$$

Step 2
$$\begin{array}{r|l} & 8 \\ \hline & 75\ 69 \\ & 64 \\ \hline & 11\ 69 \end{array}$$

Step 3
$$\begin{array}{r|l} & 8 \\ \hline & 75\ 69 \\ & 64 \\ \hline 160 & 11\ 69 \end{array}$$

Step 4
$$\begin{array}{r|l} & 8\ \ 7 \\ \hline & 75\ 69 \\ & 64 \\ \hline \cancel{160} & 11\ 69 \\ 167 & \end{array}$$

Step 5
$$\begin{array}{r|l} & 8\ \ 7 \\ \hline & 75\ 69 \\ & 64 \\ \hline \cancel{160} & 11\ 69 \\ 167 & 11\ 69 \\ \hline \end{array}$$

Explanation. 1. The largest perfect square less than 75 is 64. Write 64 below 75. Write $\sqrt{64}$, or 8, above 5 of 75.

2. Subtract 64 from 75. Bring down the next group, 69.

3. *Form the trial divisor:* $2 \times 8 = 16$; annex 0 (160).

4. *Obtain the complete divisor:* $1169 \div 160 = 7^+$. Add 7 to 160, forming 167 as the complete divisor. Write 7 over 9 of 1169.

5. Find 7×167, or 1169. Subtract. Since the remainder is zero, $\sqrt{7569} = 87$. The complete solution appears as Step 5. *It alone would be written in an actual solution.*

EXAMPLE 2. Find $\sqrt{1866.24}$.

SOLUTION. 1. *First trial divisor:* $2 \times 4 = 8$. Annex 0, giving 80.

2. *First complete divisor:* $266 \div 80 = 3^+$; $80 + 3 = 83$, the complete divisor; write 3 over the right-hand 6 of 66.

3. Place a decimal point in the square root, above that of 1866.24.

4. *Second trial divisor.* $2 \times 43 = 86$. Annex 0 (860).

5. *Second complete divisor.* $1724 \div 860 = 2^+$; $860 + 2 = 862$. Place 2 above 4 of 24. $2 \times 862 = 1724$.

$$\begin{array}{r|l} & 4\ \ 3.\ 2 \\ \hline & 18\ 66.24 \\ & 16 \\ \hline \cancel{80} & \ \ 2\ 66 \\ 83 & \ \ 2\ 49 \\ \hline \cancel{860} & 17\ 24 \\ 862 & 17\ 24 \end{array}$$

Check. We find that $(43.2)^2 = 1866.24$, or $\sqrt{1866.24} = 43.2$.

EXAMPLE 3. Find $\sqrt{645.16}$.

SOLUTION. 1. The largest perfect square less than 6 is 4 or 2^2. Hence, 2 is the first digit of the square root.

2. After forming the trial divisor, 40, it appears that the next figure may be 6 since $245 \div 40 = 6^+$. But $6 \times 46 = 276$, and this is more than 245. Therefore, we must use 5 as the second figure of the square root.

3. To form 500, we take 2×25 and annex 0. $2016 \div 500$ is 4^+. Then $500 + 4 = 504$. The result is 25.4.

$$
\begin{array}{r|l}
 & 2\ \ 5.\ 4 \\ \hline
 & 6\,45.16 \\
 & 4 \\ \hline
\cancel{40} & 2\,45 \\
45 & 2\,25 \\ \hline
\cancel{500} & 20\,16 \\
504 & 20\,16 \\ \hline
\end{array}
$$

SUMMARY. *To find the square root of a number, written in decimal notation:*

1. *Separate the number into groups (or periods) of two figures each, both ways from the decimal point.*

2. *Below the first group, write the largest perfect square less than that group. Above the group, write the square root of this perfect square.*

3. *Subtract the perfect square from the first group; bring down the next group, thus forming the first remainder.*

4. *Form the trial divisor by doubling the part of the root now found, and annexing zero. Divide the first remainder by this trial divisor, taking as the quotient only the whole number obtained. (Possibly reduce the number by 1.) Write this quotient above the next group.*

5. *Form the complete divisor by adding to the trial divisor the new figure of the square root found in Step 4.*

6. *Multiply the complete divisor by the new figure of the square root. Write the product under the remainder. Subtract.*

7. *Continue in this way, following Steps 4 to 6, until the remainder is zero, or until you have as many places in the square root as are requested.*

Note 1. As a rule, for a random number N, $\sqrt{N}$ will not be a terminating decimal. Then, in finding $\sqrt{N}$, we annex zeros at the right in N and carry out the square root process to as many decimal places as desired.

EXERCISE 104

Find the square root of each number. Obtain the result correct to hundredths by carrying out the process to thousandths.

1. 3969.	2. 134.56.	3. 273,529.	4. 8299.21.
5. 105,625.	6. 936.36.	7. 40.8321.	8. 2.1904.
9. 78.354.	10. 15,765.	11. 1643.8.	12. 7.809.

ANSWERS TO EXERCISES

Note. Answers to odd-numbered problems are given here. Answers to even-numbered problems are furnished free in a separate pamphlet when requested by the instructor.

Exercise 2. Page 7

1. 56. **3.** -12. **5.** 36. **7.** 0. **9.** -8. **11.** 56.
13. 5. **15.** -8. **17.** 4. **19.** 3. **21.** -9. **23.** -168.
25. 120. **27.** 120. **29.** -360. **31.** -2. **33.** -4. **35.** 2.
37. -13. **39.** 5. **41.** 52. **43.** $\frac{3}{4}$. **45.** -60.
47. 14.4. **49.** 4. **51.** 31. **53.** -96. **55.** 360.

Exercise 3. Page 12

1. 27. **3.** -18. **5.** 13. **7.** -16. **9.** -32.
11. -35. **13.** 0. **15.** -13. **17.** -18.2. **19.** -7.
21. 13. **23.** 61; 29. **25.** -30; 4. **27.** -36; -70.
29. 17; -17. **31.** 3.3; -11.9. **33.** -5. **35.** -7.
37. 10. **39.** -10. **41.** 22. **43.** 3. **45.** 13.
47. 42. **49.** 12. **51.** 28; 4. **53.** -40; -26.
55. 44; -32. **57.** 100. **59.** -53. **61.** -36.3.
63. -14; 14; 0; 0. **65.** 3; 9; -18; -2. **67.** 23; -23; 0; 0.

Exercise 4. Page 15

11. $<$. **13.** $<$. **15.** $>$. **17.** $>$. **19.** $<$. **21.** $>$.
29. $-5 < -3$; $|-5| > |-3|$. **31.** $0 > -3$; $|-3| > |0|$.
33. $|-2| < 7$; $-2 < 7$. **35.** $2 > -6$; $|2| < |-6|$.

Exercise 5. Page 17

1. 10. **3.** -24. **5.** 5. **7.** 8. **9.** 44.
11. 36. **13.** -2. **15.** 0. **17.** -24. **19.** -13.
21. $-2a + 5b - c$. **23.** $31 - 5a + y$. **25.** $-8a + 3b + c$. **27.** $15a$.
29. $15a$. **31.** $8a - 12$. **33.** $-15 + 5a + 30c$.
35. $18 - 12a + 15b$. **37.** $-(5 - 7a + 4b)$. **39.** $-(-6 + 3x + 4y)$.
41. $16 - (4a + b - 3c)$. **43.** $2ac - (-3 + 5a - 4c)$.

Exercise 6. Page 20

1. $11a$. **3.** $-18x$. **5.** $6cd$. **7.** $5x - 6a$.
9. $3a - 11c$. **11.** $19c - 18cd$. **13.** $-2a - 2b + 4$; $-6a + 16b - 10$.
15. $-x + ab - 4c$; $7x - 11ab + 2c$. **17.** $-3m - k - 6h$; $9m - 9k + 4h$.
19. $-14x - 3y$. **21.** $-9ac - 7xy + 4b$. **23.** $9a - 20b$.

25. $3a + 14h - 23k$. **27.** $3a + 10y - 3$. **29.** $21a - 31y + 9$.
31. $2t - 3$. **33.** $2a$. **35.** $-r - s$. **37.** 2. **39.** $13 - 4x$.
41. $-5b + 10$. **43.** $-h + 12k - 36$.

Exercise 7. Page 24

1. $\frac{3}{5}$. **3.** $\frac{15}{7}$. **5.** $\frac{1}{7}$. **7.** $\frac{1}{4}$. **9.** $\frac{7}{9}$.
11. $-\frac{5}{3}$. **13.** $\frac{3}{2}$. **15.** 3. **17.** $3/4y$. **19.** $9/2b$.
21. $-\frac{4}{3}$. **23.** $-5a/3$. **25.** $\frac{3}{10}$. **27.** $4b/d$. **29.** $\frac{5}{14}$.
31. $\frac{9}{4}$. **33.** 6. **35.** $3c/5$. **37.** $\frac{16}{5}$. **39.** $\frac{1}{10}$.
41. $\frac{12}{7b}$. **43.** $\frac{c}{7}$. **45.** $\frac{5bc}{36}$. **47.** $\frac{2}{5}$. **49.** $\frac{10}{9d}$. **51.** $\frac{5}{14}$.
53. $\frac{h}{2k}$. **55.** 4. **57.** $\frac{50}{3}$. **59.** $\frac{7}{5}$. **61.** $\frac{2ad}{c}$. **63.** $\frac{8}{125}$. **65.** $-\frac{1}{8}$.

Exercise 8. Page 27

1. 16. **3.** 100. **5.** $10,000$. **7.** 1. **9.** 1.
11. -27. **13.** -125. **15.** -243. **17.** -1000. **19.** $\frac{27}{64}$.
21. $-\frac{1}{8}$. **23.** -16. **25.** -216. **27.** -75. **29.** 320.
31. n odd, neg.; n even, pos. **33.** -120.
37. z^7. **39.** y^{10}. **41.** x^7. **43.** b^{13} **45.** a^{h+k}.
47. a^8. **49.** x^4y^4. **51.** $9x^2$. **53.** h^6. **55.** $81b^4$.
57. $\frac{h^4}{k^4}$. **59.** $\frac{x^4}{16}$. **61.** $\frac{27}{125}$. **63.** $-\frac{1}{243}$.
65. $\frac{16h^4}{k^4w^4}$. **67.** $\frac{a^4c^4}{16b^4}$. **69.** $27c^6$. **71.** $81w^4x^{12}$.
73. $81x^8$. **75.** a^8x^{12}. **77.** $\frac{c^6x^3}{125a^3}$. **79.** $\frac{x^8y^{12}}{16a^4}$.

Exercise 9. Page 28

1. $10x^6$. **3.** $3a^2b$. **5.** $-15z^8$. **7.** $4cd^5$.
9. $-4s^2t$. **11.** $-2x^2y^2$. **13.** $2a^3x^5$. **15.** $-8a^8b^3$.
17. $24r^3h^5$. **19.** $12x - 3y$. **21.** $20a - 15$. **23.** $15x + 12y$.
25. $-6x^2 - 10x^4$. **27.** $6x^2 - 2x^6$. **29.** $3h^2k - 3hk$.
31. $15w^2 - 20w^3 - 10w$. **33.** $6a^4b^9$. **35.** $24m^5n^4$.
37. $6x^{h+n}y^{1+k}$. **39.** $-3h^{4+r}k^{3+s}$.
41. $14x^5 - 10x^4 + 8x^3 - 12x^2$. **43.** $3a^4b^4 + 15a^3b^3 + 18a^2b^2 - 9ab$.
45. $3 - 2y + 4y^2$. **47.** $4a^2 - 10a - 14$. **49.** $6x - 24x^2 - 12$.

Exercise 10. Page 30

1. $x^2 + x - 12$. **3.** $2x^2 - 3x - 35$. **5.** $20a^2 - 43a + 21$
7. $4h^2 - 9k^2$. **9.** $2a^2 + ab - 15b^2$. **11.** $9r^2 - 25s^2$.
13. $2a^2b^2 - ab - 15$. **15.** $c^2d^2 - x^2$. **17.** $a^2 + 6a + 9$.
19. $h^2 - 8hk + 16k^2$. **21.** $9a^2 - 12ab + 4b^2$. **23.** $a^2x^2 - 2abx + b^2$

25. $y^6 + 3y^3 - 10.$ **27.** $3a^6 + a^3b^2 - 4b^4.$ **29.** $y^3 - 8.$
31. $x^3 - x^2 - 11x + 15.$ **33.** $6 - 5x - 6x^2 - x^3.$ **35.** $2x^3 - 5x^2 - 8x + 5.$
37. $20 - 14b - 3b^2 + b^3.$ **39.** $6x^4 - 7x^3 + 12x^2 - 19x + 7.$
41. $2y^4 - 20y^3 - 6y^2 + 25y - 25.$ **43.** $15x^4 - 17x^3 + 12x^2 + 17x - 15.$
45. $25x^2 + 4y^2 + 9 + 30x - 12y - 20xy.$ **47.** $a^3 + b^3.$
49. $2x^3 - x^2 - 16x + 15.$ **51.** $6a^3 - 11a^2 - 17a + 30.$ **53.** $x^{3n} - 27y^{3k}.$

Exercise 11. Page 33

1. $y^2.$ **3.** $1/x^5.$ **5.** $x.$ **7.** $9.$ **9.** $1/x.$
11. $2y^2.$ **13.** $4x.$ **15.** $7a.$ **17.** $1/5r.$ **19.** $\frac{z^4}{3}.$
21. $\frac{7}{a^3}.$ **23.** $\frac{x^3}{y^3}.$ **25.** $\frac{k^3}{h}.$ **27.** $9st.$ **29.** $-\frac{6b}{c^2}.$
31. $\frac{1}{4c}.$ **33.** $7x.$ **35.** $-\frac{1}{3}.$ **37.** $-\frac{1}{3b^2}.$ **39.** $-6xy.$
41. $\frac{3}{2}a + 5b.$ **43.** $-a - 4b.$ **45.** $3a - 2a^2.$ **47.** $-2 + 5a^2.$
49. $\frac{4}{h} - \frac{8}{h^2}.$ **51.** $x^2 - 2x + 3.$ **53.** $y^2 - y + 5.$
55. $\frac{1}{15} - \frac{1}{5x^2} + \frac{1}{3x^3}.$ **57.** $6b - 3a.$ **59.** $-\frac{7x^2}{2} + 2x + \frac{3}{2} - \frac{1}{x}.$
61. $\frac{2x}{y} - 4y + \frac{y^3}{3x}.$ **63.** $d^2 - a - \frac{a^2}{bd}.$

Exercise 12. Page 36

1. $x + 4.$ **3.** $c - 3.$ **5.** $s - 3.$ **7.** $y - 4.$
9. $2c - 3 - \frac{6}{2c + 3}.$ **11.** $x^2 + 4.$ **13.** $2x + 3 + \frac{2}{x + 2}.$
15. $a - b - \frac{5b^2}{2a + b}.$ **17.** $x^3 + 2.$ **19.** $3x + 11 + \frac{19}{x - 2}.$
21. $x^2 - 2.$ **23.** $2x^2 - x - 6 - \frac{11}{2x - 3}.$ **25.** $2y - 3 - \frac{2y}{4y^2 - 3y + 2}.$
27. $x^3 - x^2 - 4 + \frac{3}{x - 3}.$ **29.** $x^2 + 3xy + 4y^2.$ **31.** $x^2 - 3x + 9.$
33. $x^2 + xy + y^2.$ **35.** $4w^2 + 6w + 9.$ **37.** $a^4 + a^2b^2 + b^4.$ **39.** $2x^2 - 3z.$

Exercise 13. Page 39

1. $\frac{1}{4}.$ **3.** $\frac{2 + b - a}{3}.$ **5.** $\frac{5 - b}{a}.$
7. $\frac{3 - 2a + 5b}{7}.$ **9.** $\frac{2y - x - 7}{8}.$ **11.** $\frac{1 - a}{b}.$
13. $-\frac{5x}{a^2b}.$ **15.** $\frac{cd + 1}{x^2y}.$ **17.** $6.$ **19.** $34.$
21. $3bc.$ **23.** $2a^2b^2.$ **25.** $28hx^3.$ **27.** $\frac{39}{32}.$

29. $\dfrac{5c}{35}$. **31.** $\dfrac{3x}{6y}$. **33.** $\dfrac{6axy^3}{18x^3y^7}$. **35.** $\dfrac{12ab^2d}{20a^5b^3}$. **37.** 48.
39. 300. **41.** 18,900. **43.** $24a^3b^6$. **45.** $36a^3x^4$. **47.** $80h^3k^3$.

Exercise 14. Page 41

1. $\frac{11}{8}$. **3.** $\frac{1}{2}$. **5.** $\frac{7}{16}$. **7.** $\frac{11}{21}$. **9.** $\dfrac{2a+b}{6}$.
11. $\dfrac{4h-3k}{12}$. **13.** $\frac{31}{40}$. **15.** $-\frac{4}{3}$. **17.** $\frac{13}{15}$. **19.** $\dfrac{10h-3}{14}$.
21. $\dfrac{2-3x+6y}{9}$. **23.** $\dfrac{6-5x+y}{3}$. **25.** $\dfrac{5a+2b-11}{10}$.
27. $\dfrac{14a-57}{25}$. **29.** $-\dfrac{1}{12a}$. **31.** $\dfrac{3h-4w}{12k}$.
33. $-\dfrac{13}{12y}$. **35.** $\dfrac{5r-3h}{hkr}$. **37.** $\dfrac{10b^2-3a}{2a^2b^3}$.
39. $\dfrac{20b-15ay+3a}{15a^2b}$. **41.** $-\dfrac{19a+94}{20}$. **43.** $\dfrac{6-x}{6x^2}$.
45. $\dfrac{3y^2-10y+12}{4y^3}$. **47.** $\dfrac{5x-9}{6x}$. **49.** $\dfrac{2y^2z-4yz-6y+9}{6y^2z^2}$.
51. $\dfrac{3-13x}{42}$. **53.** $\dfrac{27y-10y^2+20}{20y^3}$. **55.** $\dfrac{3-8ab^3-4a^2b^4}{4a^2b^3}$.

Exercise 15. Page 44

1. $\frac{391}{15}$. **3.** $\frac{209}{21}$. **5.** $\dfrac{6-103b+420b^2}{35}$. **7.** $\frac{215}{8}$.
9. $\frac{19}{44}$. **11.** $\frac{31}{55}$. **13.** $\dfrac{45a-6}{30a-5}$.
15. $\dfrac{2-3a}{5+4a}$. **17.** $\dfrac{5-3bc}{4bc+3}$. **19.** $\dfrac{18y-20x}{24y+9x}$.
21. $\dfrac{5a^2b-3}{7a^2b-2}$. **23.** $\dfrac{15y^2-18x^2y}{4x^2-3xy^2}$. **25.** $\dfrac{25x-30wx^2}{20x^2-6}$.
27. $\dfrac{5a+3ab}{2a+3}$. **29.** $\dfrac{4x+3xy}{2-3x}$. **31.** $\dfrac{8a+15b}{36x-24}$.
33. $\frac{1}{75}$. **35.** $\frac{3}{8}$. **37.** $-\frac{3}{2}$. **39.** $\frac{5}{63}$. **41.** $-\frac{3}{16}$.
43. $\dfrac{1}{3b-2a}$. **45.** $\dfrac{2a-3h}{3a+5h}$.

Exercise 16. Page 46

1. -60. **3.** 0. **5.** $\frac{5}{9}$. **7.** -4. **9.** 1. **11.** 6.
13. 17. **15.** -32; -14. **17.** 42; -8. **19.** $<$. **21.** $>$.
23. $2b+c-3a$. **25.** $a^3b^2-3a^5b$. **27.** $6x^3y^3-12x^4y^4$.
29. $9k-11h$. **31.** $15-2a$. **33.** $-\frac{5}{7}$. **35.** $\frac{8}{15}$.
37. $\frac{51}{7}$. **39.** $\frac{28}{3}$. **41.** $24h^6k^7$. **43.** $81x^8y^4$.

45. $625c^8d^{12}y^4$. 47. $\frac{27}{125}$. 49. $\frac{a^2b^2}{4}$. 51. $\frac{9a^2}{4x^2}$. 53. $\frac{81}{x^{12}}$.
55. $4x^2 - 8x - 21$. 57. $6x^2 - 13xy + 15y^2$.
59. $2x^3 - 5x^2 - 8x^4 - 13x + 15$. 61. a^4. 63. $-\frac{d}{4c^3}$.
65. $2x^2 + 4x - 3 + \frac{3}{2x - 5}$. 67. $-\frac{4 + 8y^6}{2y^4}$. 69. $\frac{5}{3}$.
71. $\frac{31 - 12a}{12}$. 73. $\frac{6y^2 - 9xy^2 - 20x + 12xy}{12x^2y^3}$. 75. $\frac{19}{21}$.
77. $\frac{9y^2 - 30x^2y}{4x^2 - 3xy^2}$. 79. $\frac{2xy - 3x^2y}{3y - 5x}$. 81. $\frac{5}{3}$; $\frac{3}{a - 5}$.

Exercise 17. Page 50

1. 3.25. 3. 100,000. 5. .0001. 7. .0000001.
9. $3(10^3) + 10^2 + 4(10) + 9$. 11. $\frac{3}{10} + \frac{1}{10^2} + \frac{9}{10^3}$.
13. 536.437. 15. 5735.35. 17. 14.1192. 19. .0681.
21. 6.64; 3.88. 23. − 103.7698; 16.0762. 25. 1.178.
27. 5.32. 29. 326,530. 31. .000317. 33. 5.738.

Exercise 18. Page 52

1. 4.914. 3. 5.993. 5. .51312. 7. 13.62528.
9. .000054322. 11. 2.1435402.

Exercise 19. Page 55

1. 15.326; 15.3. 3. .31486; .315. 5. 195.64; 196.
7. .034564; .0346. 9. 566.5 and 567.5. 11. 567.35 and 567.45.
13. 31.54; .586. 15. 11.4034; .054. 17. 2738.
19. 2,056,000. 21. $10^2(6.7538)$. 23. $4.5726(10^4)$.
25. $4.5312(10^5)$; $4.53(10^5)$. 27. $7.2200(10^7)$; $7.22(10^7)$. 29. $2.6(10^3)$ cu. ft

Exercise 20. Page 57

1. 1.37. 3. 57.2. 5. .263. 7. 150. 9. .02981. 11. .286.
13. $\frac{11}{4}$. 15. $\frac{19}{8}$. 17. $\frac{13}{400}$. 19. .625. 21. $.\dot{1}\dot{5}$. 23. .4375.

Exercise 21. Page 62

1. $2\frac{1}{2}$. 3. − 3. 5. $\frac{1}{2}$. 7. 0. 9. $-\frac{7}{3}$
11. $-\frac{13}{9}$. 13. 1. 15. $\frac{8}{7}$. 17. $\frac{5}{4}$. 19. $-\frac{2}{3}$.
21. $\frac{7}{3}$. 23. − 3. 25. $\frac{12}{5}$. 27. 15. 29. $\frac{2}{3}$.
31. 8. 33. 6. 35. 2. 37. 4. 39. 17.
41. 3. 43. .36. 45. $\frac{2}{3}$. 47. 4. 49. $\frac{3}{2}$.
51. $\frac{29}{17}$. 53. − 7. 55. 2. 57. $-\frac{3}{2}$. 59. 283.46. 61. 515.02.

Exercise 22. Page 66

1. $\frac{3 + c}{b}$. 3. $\frac{3h + 5a}{c}$. 5. $\frac{2b}{3 - a}$. 7. $\frac{4a + 5c}{2a - 3b}$.
9. $\frac{5c}{b - 2a}$. 11. $\frac{ab}{2}$. 13. $\frac{cd}{2}$. 15. $\frac{2ab}{3c}$.
17. $\frac{2h}{3k}$. 19. $\frac{ABC}{12}$. 21. $\frac{4ab}{3b - a}$. 23. $\frac{2(ab - abc)}{a^2 - bd}$.
25. $\frac{2b + 9ab - 15}{6a - 5c}$. 27. 2. 29. $-\frac{2}{35}$.
31. $C = \frac{5F - 160}{9}$; (*a*) 0°; (*b*) 100°; (*c*) 26.7°; (*d*) 10°. 33. $a = \frac{f}{m}$.
35. $a = l - (n - 1)d$; $n = \frac{l - a + d}{d}$; $d = \frac{l - a}{n - 1}$.
37. $a = \frac{rS - S}{r^n - 1}$. 39. $A = \frac{M + N + P}{3}$. 41. $C = .12n + 6$.

Exercise 23. Page 69

1. 32.5″ and 35.5″. 3. 22.5′ and 5′.
5. $3\frac{3}{5}$. 7. 15; 16; 17. 9. 8′. 11. 40′; 120′.
13. 13 nickels; 39 dimes; 36 quarters. 15. 80 bu.
17. $8\frac{2}{5}$ hr. 19. $3\frac{15}{16}$ da. 21. $2\frac{10}{13}$ hr.

Exercise 24. Page 72

1. .05. 3. .0375. 5. 1.263. 7. 7%. 9. $2\frac{1}{2}$%. 11. 135%.
13. 8.32. 15. $37\frac{1}{2}$% of 200. 17. 175% of 200. 19. 452.9, approximately.
21. 560 dimes. 23. $22,000. 25. 75 lb. at 70¢; 25 lb. at 50¢.
27. 20 gal. 29. Approximately 88.9 bu. at $1.25 and 111.1 bu. at $.80.
31. 3 gal. 33. $58\frac{1}{3}$%.

Exercise 25. Page 74

1. $21\frac{9}{11}$ ft. from fulcrum on other side. 3. $63\frac{1}{3}$ lb.
5. 8 ft. from fulcrum on side of 40 lb. weight. 7. $69\frac{3}{13}$ lb.

Exercise 26. Page 77

1. 50 m.p.h. 3. At end $7\frac{1}{7}$ hr. 5. $16\frac{1}{5}$ sec.
7. $\frac{ty}{x}$ sec. 9. 310 m.p.h. 11. At end 10 yr.
13. 1792 mi.; 7 hr. and 28 min. 15. $1306\frac{2}{3}$ mi.; 7 hr. and 28 min.
17. Approximately 8.13 hr. 19. At $10\frac{10}{11}$ min. after 2 P.M.

Exercise 27. Page 80

1. $180.00; $5180.00. 3. $48.00; $3048.00.
5. $159.00. 7. $2914.98. 9. $42,857.14.

11. \$1000.00. **13.** \$5000.00. **15.** At 5%; gains \$17.86.
17. \$4000 at 5%; \$3000 at 4%.

Exercise 28. Page 83

1. ± 5. **3.** ± 11. **5.** $\pm \frac{1}{3}$. **7.** 3. **9.** 9.
11. 14. **13.** $\frac{4}{5}$. **15.** $\frac{1}{9}$. **17.** $\frac{2}{9}$. **19.** x^2.
21. a. **23.** a^6. **25.** $2a^2$. **27.** $2z^4$. **29.** $7z^3$.
31. $8w^3x^2$. **33.** $7w^2x^2$. **35.** $\dfrac{3}{a}$. **37.** $\dfrac{y}{4}$. **39.** $\dfrac{7}{w^2}$. **41.** $\dfrac{a}{y^3}$.
43. $\dfrac{2x}{h^4}$. **45.** $\dfrac{9a}{yz}$. **47.** $\dfrac{3ab^2}{c^3w^5}$. **49.** $\dfrac{11a}{3b^2z^3}$. **51.** 37. **53.** yz^3.

Exercise 29. Page 86

1. $15a - 20v$. **3.** $4abx - a^2bx$. **5.** $c^2 - d^2$.
7. $a^2 + 2ay + y^2$. **9.** $16 - y^2$. **11.** $9 - 4r^2$.
13. $a^4 - 9b^2$. **15.** $a^2 - 6a + 8$. **17.** $x^2 + 10x + 25$.
19. $4a^2 - 20a + 25$. **21.** $4z^2 - 4wz + w^2$. **23.** $4a^2 + 4ab + b^2$.
25. $6 + 5x + x^2$. **27.** $x^2 + 4x - 45$. **29.** $a^2 + 5ab + 6b^2$.
31. $6x^2 + 17x + 12$. **33.** $8y^2 - 10xy + 3x^2$. **35.** $6y^2 + y - 15$.
37. $21w^2 + 29w - 10$. **39.** $8x^2 + 6xy - 9y^2$. **41.** $-12 + 16x - 5x^2$.
43. $-3x^2 + 19x - 20$. **45.** $x^4 + 4x^2 + 4$. **47.** $4x^2y^2 - 12xy^3 + 9y^4$.
49. $9 + 24bx + 16b^2x^2$. **51.** $x^2 - \frac{2}{3}x + \frac{1}{9}$. **53.** $\frac{1}{9} - \frac{4}{3}z + 4z^2$.
55. $c^6d^2 - 9x^4$. **57.** $x^2 + .3x - .1$. **59.** $6 + 1.1x - .1x^2$.
61. $\frac{1}{4}a^2 - \frac{1}{9}b^2$. **63.** $.08x^2 - .26x - .15$. **65.** $16 - 8x^4 + x^8$.
67. $-14ax^2 + 21x^3 + 7x^4$. **69.** $6 - 7x - 20x^2$.
71. $4x^2 + 12xy + 9y^2$. **73.** $4x^2 - 8xy + 4y^2$.
75. $100c^2 - 300cd + 225d^2$. **77.** $12x^4 - x^2 - 6$.
79. $2a^4 - a^2b^2 - 15b^4$. **81.** $21a^6 + a^3b^3 - 10b^6$.
83. $12x^6 - x^3y^4 - 6y^8$. **85.** $9u^4 - 15u^2v^2 - 14v^4$.

Exercise 30. Page 88

1. $x^2 + y^2 + 2xy + 4 + 4x + 4y$. **3.** $9 - 12x + 6y + 4x^2 - 4xy + y^2$.
5. $9x^2 + y^2 + 25 + 6xy + 10y + 30x$. **7.** $16a^2 + b^2 + c^2 - 8ab - 8ac + 2bc$.
9. $4x^2 - 12ax + 12b^2x + 9a^2 + 9b^4 - 18ab^2$.
11. $x^2 + 2xy + y^2 - 9$. **13.** $16 - 4a^2 - 4ab - b^2$. **15.** $a^2 + 2ab + b^2 - x^2$.
17. $9a^2 - 6ay + y^2 - 16$. **19.** $x^4 - y^2 + 2yz - z^2$.
21. $4x^2 + 4xy + y^2 + a^2 - 6a + 9 + 4ax - 12x + 2ay - 6y$.
23. $4x^2 + z^2 - 4xz + y^2 - 4y + 4 + 4xy - 8x - 2yz + 4z$.
25. $4x^2 + 4xy + y^2 - z^2 + 6z - 9$. **27.** $c^2 - 4cd + 4d^2 - a^2 - 2ax - x^2$.
29. $4a^2 + 9b^2 + 16c^2 + 12ab + 16ac + 24bc$.
31. $w^2 + 25x^2 + 9a^2 - 10wx + 6aw - 30ax$.

Exercise 31. Page 90

1. $x(3 + b)$. **3.** $2x(3y^2 + a)$. **5.** $y(2c + d^3 + 1)$.
7. $x(3b - a + c)$. **9.** $t(t - ct^2 - 4a)$.

11. $ay^2(3ay - 2 + ay^2)$. **13.** $w^2x^2(2w^2x - 6 + 5wx^2)$.
15. $(w - z)(w + z)$. **17.** $(8 - xy)(8 + xy)$. **19.** $(2x - y)(2x + y)$.
21. $(6d + 11)(6d - 11)$. **23.** $(2a + 3b)(2a - 3b)$. **25.** $(16a + 1)(16a - 1)$.
27. $(\frac{1}{3} + w)(\frac{1}{3} - w)$. **29.** $(5w + cd)(5w - cd)$.
31. $(6ab + 8x)(6ab - 8x)$. **33.** $a(x - y)(x + y)(x^2 + y^2)$.
41. $(x + b)^2$. **43.** $(a - 1)^2$. **45.** $12w$; $(w - 6)^2$.
47. $(x - 9)^2$. **49.** $(7x + a)^2$. **51.** $(8 - ab)^2$.
53. $12xz$; $(2x + 3z)^2$. **55.** $20acd$; $(2cd - 5a)^2$.
57. $(3x - 5y)^2$. **59.** $(2x^2 - 7)^2$. **61.** $(2a^2 - 3b^2)^2$.
63. $(7z - 2b)(7z + 2b)$. **65.** $4u(3v - w)(3v + w)$.
67. $(x - 5y^2)^2$. **69.** $x(2a - 1)^2$ **71.** $2(7u^2 - 5)(7u^2 + 5)$
73. $25(x - 2b^2)(x + 2b^2)$. **75.** $(4x^2 + 25v^2)(2x - 5v)(2x + 5v)$.
77. $2(3u - 5v)^2$. **79.** $3(7x - 5v^2w^2)(7x + 5v^2w^2)$.
81. 4(100). **83.** 1600. **85.** 280.

Exercise 32. Page 93

1. $(x + 5)(x + 3)$. **3.** $(a - 6)(a - 2)$. **5.** $(x - 5)(x - 3)$.
7. $(t + 7)(t - 3)$. **9.** $(x - 6)(x + 3)$. **11.** $(w - 6)(w + 8)$.
13. $(5 + w)(3 - w)$. **15.** $(6 - w)(4 + w)$. **17.** $(8 + y)(4 - y)$.
19. $(9 + k)(6 - k)$. **21.** $(x - 12)(x + 6)$. **23.** $(5a + 7)(a + 1)$.
25. $(5x - 3)(2x - 1)$. **27.** $x^2(4x - 3)(2x - 1)$. **29.** $y(3y + 5)(y - 1)$.
31. $(3x^3 - 5)(x^3 + 2)$. **33.** $(4w^3 + 3)(2w^3 - 3)$. **35.** $(5a^2 - 7)(3a^2 + 4)$.
37. $(7 + 2x)(1 - 3x)$. **39.** $(1 - 3x)(9x + 2)$. **41.** $(3x + 2y)(x + y)$.
43. $(2w + 5z)(4w - 3z)$. **45.** $(6w + u)(2w - 5u)$. **47.** $(3a - 5b)(2a - b)$.
49. $(10a + x^2)(10a - x^2)$. **51.** $(x - 2y)(x + 2y)(x^2 + 4y^2)$.
53. $(8a - 3c)^2$. **55.** $(4 - 3x)(2x + 5)$. **57.** $2x(x - y)(x + y)$.
59. $(3x^2 + 2y)^2$. **61.** $(5x + 10b^2)(5x - 10b^2)$. **63.** $r(2 - 5h)(1 - 3h)$.
65. $(\frac{1}{3} - 2y)(\frac{1}{3} + 2y)(\frac{1}{9} + 4y^2)$. **67.** Prime.
69. $(3x^2 + 5)(x^2 - 4)$. **71.** $z^4(5w - 2)(5w + 2)(25w^2 + 4)$.
73. $(2y^h + z^n)(2y^h - z^n)$. **75.** $-(3a - 5b)^2$.
77. $(2x^2 - 5)(x^2 + 3)$. **79.** $(3a^2 - 5y^2)(a^2 + 3y^2)$.

Exercise 33. Page 95

1. $2(x + 2y)$. **3.** $(c + d)(x + y)$. **5.** $(2h - 3k)(m - 2)$.
7. $(2d - 5c)(r + s)$. **9.** $(3h - 1)(w - z)$. **11.** $(3a + 2b)(w - 2k)$.
13. $(a + b)(3c + d)$. **15.** $(c + 3d)(r - s)$. **17.** $(2x + y)(c - d)$.
19. $4(x - b)(h - 2c)$. **21.** $(x - 2)(x - 1)(x + 1)$. **23.** $(x + 2)(x^2 + 1)$.
25. $(x - 3)(x^2 + 1)$. **27.** $(a - 3)(a^2 + 1)$. **29.** $(3x - 2)(x^2 + 2)$.
31. $(2 + x)(r - s)$. **33.** $(x - s - 3)(x + s + 3)$.
35. $(2z + w - y)(2z + w + y)$. **37.** $(c - 3d - 2x - y)(c - 3d + 2x + y)$.
39. $(z + 1 - 3x)(z + 1 + 3x)$. **41.** $(y + z + 2x)(y + z - 2x)$.
43. $(2a - 3z - 1)(2a + 3z + 1)$. **45.** $(3x - y + z)(3x + y - z)$.
47. $(4a - 1 + 3x)(4a + 1 - 3x)$. **49.** $(b + c)(x - y)(x + y)(x^2 + y^2)$.

51. $(z^2 - w)(z^2 + w - 1)$. 53. $(r + 3t - a - b)(r + 3t + a + b)$.
55. $(c + 2 - 3d - h)(c + 2 + 3d + h)$.
57. $(3x - y - 5a + b)(3x - y + 5a - b)$. 59. $(a + b + 3x)(a + b - 3x)$.
61. $(2a - 3b + 2x + y)(2a - 3b - 2x - y)$.
63. $(2x - 3y)(2x + 3y)(4x^2 + 9y^2 + 1)$.

Exercise 34. Page 98

1. $x^2 - xy + y^2$. 3. $a^2 - 3ab + 9b^2$. 5. $c^3 + w^3$.
7. $27a^3 - c^3$. 9. $1 - 27x^3$. 11. $b^3 - 8x^3$.
13. $(d - y)(d^2 + dy + y^2)$. 15. $(y - 3)(y^2 + 3y + 9)$.
17. $(1 - v)(1 + v + v^2)$. 19. $(z + 10)(z^2 - 10z + 100)$.
21. $(1 - 3x)(1 + 3x + 9x^2)$. 23. $(z - 2w)(z^2 + 2wz + 4w^2)$.
25. $(6x - yz)(36x^2 + 6xyz + y^2z^2)$. 27. $(7a - 2xz)(49a^2 + 14axz + 4x^2z^2)$.
29. $h^3 - 3h^2k + 3hk^2 - k^3$. 31. $u^3 + 9u + 27u^2 + 27$.
33. $8x^3 + 12wx^2 + 6w^2x + w^3$. 35. $64x^3 + 48x^2y + 12xy^2 + y^3$.
37. $a^6 - 6a^4x + 12a^2x^2 - 8x^3$. 39. $c^3 - 6b^2c^2 + 12b^4c - 8b^6$.
41. $8c^6 - 36c^4z + 54c^2z^2 - 27z^3$.
43. $(x + 2)(x - 1)(x^2 + x + 1)(x^2 - 2x + 4)$.
45. $(2x - 3y)(4x^2 + 6xy + 9y^2)(x + y)(x^2 - xy + y^2)$. 47. $(a - 1)^3$.
49. $(w - 3x)^3$. 51. $(c - d - a)(c^2 - 2cd + d^2 + ac - ad + a^2)$.

Exercise 35. Page 99

1. $(a^2 + a + 1)(a^2 - a + 1)$. 3. $(3a^2 + 2a + 1)(3a^2 - 2a + 1)$.
5. $(z^2 + hz + h^2)(z^2 - hz + h^2)$. 7. $(2w^2 + 2aw + 3a^2)(2w^2 - 2aw + 3a^2)$.
9. $(5a^2 + 5ab + 2b^2)(5a^2 - 5ab + 2b^2)$. 11. $(x^2 - 2x + 2)(x^2 + 2x + 2)$.
13. $(z^2 + 4hz + 8h^2)(z^2 - 4hz + 8h^2)$. 15. $(9z^2 + 12xz + 8x^2)(9z^2 - 12xz + 8x^2)$.
17. $(3a^2 + 2ac - 2c^2)(3a^2 - 2ac - 2c^2)$. 19. $(5a + 3y)(a - y)(5a - 3y)(a + y)$.
21. $(3x^2 + 3xy - 5y^2)(3x^2 - 3xy - 5y^2)$.

Exercise 36. Page 101

1. $(2ab)^3$. 3. $(2ab)^4$. 5. $(5x^2y)^3$. 7. $(4u^2v^3)^4$.
9. $(a - x)(a + x)(a^2 + x^2)$. 11. $(2 - w)(2 + w)(4 + w^2)$.
13. $(x^4 + y^4)(x^2 + y^2)(x - y)(x + y)$. 15. $(3 - 2x)(3 + 2x)(9 + 4x^2)$.
17. $(u - 1)(u + 1)(u^2 + u + 1)(u^2 - u + 1)$.
19. $(x - 2y)(x + 2y)(x^2 + 2xy + 4y^2)(x^2 - 2xy + 4y^2)$.
21. $(x^2 + 1)(x^4 - x^2 + 1)$. 23. $(x^2 + 9)(x^4 - 9x^2 + 81)$.
25. $(4 + a^2)(2 - a)(2 + a)(16 + a^4)$.
27. $(a + b)(a^2 - ab + b^2)(a^6 - a^3b^3 + b^6)$.
29. $(3x^2 - y)(3x^2 + y)(9x^4 + y^2)$. 31. $(5 - 2x^2)(5 + 2x^2)(25 + 4x^4)$.
33. $(a - 2b)(a + 2b)(a^2 + 2ab + 4b^2)(a^2 - 2ab + 4b^2)$.
35. $(2a - 3x^2)(4a^2 + 6ax^2 + 9x^4)$.

Exercise 37. Page 103

1. $x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5$. 3. $x^4 - 2x^3y + 4x^2y^2 - 8xy^3 + 16y^4$.
5. $a^4 - a^3y + a^2y^2 - ay^3 + y^4$.

7. $x^7 - x^6y + x^5y^2 - x^4y^3 + x^3y^4 - x^2y^5 + xy^6 - y^7.$

9. $z^4 + wz^3 + w^2z^2 + w^3z + w^4.$ **11.** $x^4 + x^3 + x^2 + x + 1.$

13. $x^9 - x^8y + x^7y^2 - x^6y^3 + x^5y^4 - x^4y^5 + x^3y^6 - x^2y^7 + xy^8 - y^9.$

15. $a^2 + 2a + 4.$ **17.** $x^4 + y^4.$ **19.** $a^6 - a^4b + a^2b^2 - b^3.$

21. $x^6 + x^3y^3 + y^6.$ **23.** $x^9 - x^6y^3 + x^3y^6 - y^9.$

25. $4x^2 + 6x + 9.$ **27.** $z^4 + 2x^2z^3 + 4x^4z^2 + 8x^6z + 16x^8.$

29. $(a - w)(a + w)(a^2 + w^2).$ **31.** $(u + w)(u^4 - u^3w + u^2w^2 - uw^3 + w^4).$

33. $(1 - y)(1 + y)(1 + y + y^2)(1 - y + y^2).$

35. $(u - v)(u^2 + uv + v^2)(u^6 + u^3v^3 + v^6).$

37. $(2a - 1)(16a^4 + 8a^3 + 4a^2 + 2a + 1).$

39. $(2 + x)(64 - 32x + 16x^2 - 8x^3 + 4x^4 - 2x^5 + x^6).$

41. $(a - 3x^2)(a^2 + 3ax^2 + 9x^4).$ **43.** Prime.

45. $(u^h + v)(u^{2h} - u^hv + v^2).$ **47.** Prime.

49. $(2 - x)(4 + 2x + x^2)(64 + 8x^3 + x^6).$ **51.** $(u^h + v^k)(u^{2h} - u^hv^k + v^{2k}).$

Exercise 38. Page 105

1. $\dfrac{3x}{2y^2}.$ **3.** $\frac{5}{3}.$ **5.** $\dfrac{c}{d^2}.$ **7.** $\dfrac{x + y}{2}.$ **9.** $\dfrac{5x + 10}{2}.$

11. $\dfrac{2a + 3b}{x}.$ **13.** $\dfrac{m + 6}{m + 4}.$ **15.** $\dfrac{x + 5}{3x + 2}.$ **17.** $\dfrac{2x - 4a}{x + 3a}.$

19. $\dfrac{2}{x - 2y}.$ **21.** $\dfrac{a^2 + ab + b^2}{2}.$ **23.** $\dfrac{9x^2 + 6xy + 4y^2}{3x + 2y}.$

25. $-\dfrac{2x + 2y}{a + b}.$ **27.** $-\dfrac{2}{x + y}.$ **29.** $\dfrac{c - d}{c}.$

31. $\dfrac{3 - x}{6 + 2x}.$ **33.** $-\dfrac{x + 3}{6 + 4x}.$ **35.** $\dfrac{5 - 2x}{x^2 + 3x + 9}.$

37. $\dfrac{(2x + 5)(2x + 1)}{3d(2x - 1)}.$ **39.** $\dfrac{x^2 - 9b^2}{2b - x}.$

Exercise 39. Page 107

1. $\dfrac{3x^2 + 9x}{(x + 3)(x - 2)}.$ **3.** $\dfrac{6x^2 + 9x}{4x^2 - 9}.$ **5.** $\dfrac{6 - 2a}{2a - 4}.$

7. $\dfrac{10bx - 3ax}{4ab}.$ **9.** $\dfrac{11}{15(a - b)}.$ **11.** $\dfrac{19}{15(x - y)}.$

13. $\dfrac{6x^2 - 5xy + 6y^2}{9x^2 - 4y^2}.$ **15.** $\dfrac{1}{6d - 2c}.$ **17.** $\dfrac{11x + 4}{3(4x^2 - 1)}.$

19. $\dfrac{13x - 2x^2 + 10}{6x^2 + 6x}.$ **21.** $\dfrac{19x + 4}{(2x - 1)(3x + 3)}.$ **23.** $\dfrac{6a^2 - 6}{2a - 3}.$

25. $\dfrac{3x^2 - 3xy + 8}{2x^2 - 2y^2}.$ **27.** $\dfrac{3a + n}{6(a - n)}.$ **29.** $\dfrac{x^2 - 17x + 1}{x^2 + x - 12}.$

31. $\dfrac{14 + 2n}{3(1 - n)(n + 4)}.$ **33.** $\dfrac{6c^2 - 5c + 30}{2(3c - 2)(c^2 - 9)}.$ **35.** $\dfrac{16x^2 + 36x + 45}{(3 - 2x)(8x^3 - 27)}.$

37. $\dfrac{11x^4 - 22x^2 + 6}{(x^4 - 4)(2x^2 - 3)}$. **39.** $\dfrac{12x^3 + 11x^2 - 25x - 9}{3(x + 1)^2(2x - 3)}$. **41.** $\dfrac{13x^2 + 18x - 3}{2(9 - 4x^2)(x - 3)}$.

43. $\dfrac{9ar - r^2 - 6a^2 + 81a^4 - 9a^2r^2 + 3ar^3 - 27a^3r}{(r - 3a)^2(r + 3a)}$.

Exercise 40. Page 110

1. $\frac{3}{2}$. **3.** $-\dfrac{hw}{3a}$. **5.** $\dfrac{(x - 1)(x + 4)}{x}$.

7. $\dfrac{y(3y + 1)}{2y - 8}$. **9.** $\dfrac{x^2 + 3x}{5 + 3x}$. **11.** $\dfrac{(x - 2y)(x + 3)}{2x}$.

13. $\dfrac{(2a + 3b)(a^2 - ab + b^2)}{2}$. **15.** $\dfrac{3x^2}{b - a}$.

17. $\dfrac{a - 2b}{a + 3b}$. **19.** $\dfrac{a}{12}$. **21.** $\dfrac{xy}{2y + 3x}$.

23. $\dfrac{2y}{10xy - 1}$. **25.** $\dfrac{u^2 + 2u + 4}{u^2}$. **27.** $\dfrac{2}{a^2 + 1}$.

29. $\dfrac{3x + 4}{x - 2}$. **31.** $\dfrac{2y(y^2 + xy + x^2)}{x^2}$. **33.** $\dfrac{1}{2x(x + 4)}$.

35. $\dfrac{1}{c(n - v)}$. **37.** $\dfrac{y(y + 2)}{y + 5}$. **39.** $\dfrac{6ab}{2a + 5b}$.

41. $\dfrac{(a + 3b)(a^2 + 9b^2)}{ac - 3bc}$. **43.** $\dfrac{2a - 3b}{a^2b^2}$. **45.** $\dfrac{z + 4a}{z - 4a}$.

47. $\dfrac{(4x^2 + 9y^2)(2x + 3y)}{3a^2b(x - y)}$. **49.** $\dfrac{2x + 2}{2 - x}$. **51.** $\dfrac{a^4b^2(a^2 - b^2)}{3a^2 - 2b^2}$.

53. $\dfrac{5}{a + 3}$. **55.** $\dfrac{(13a - 3)(1 + a)}{2a(5a - 1)}$. **57.** $\dfrac{(3a - 2)(2a - 1)(a + 3)}{(5a - 3)(3 - 3a)}$.

Exercise 41. Page 115

1. 14. **3.** -5. **5.** $\frac{1}{4}$. **7.** -11. **9.** 3.
11. 2. **13.** -2. **15.** 5. **17.** 4. **19.** $-\frac{8}{7}$.
21. $\frac{12}{35}$. **23.** -5. **25.** 1. **27.** 3. **29.** $\frac{12}{17}$.
31. 4 hr. **33.** 380 m.p.h. **35.** 15 m.p.h.

Exercise 42. Page 117

1. $\dfrac{2h + 3a}{c}$. **3.** $3a + b$. **5.** $\dfrac{a + 1}{4}$. **7.** $2n$.

9. $\dfrac{a^2 + b^2}{2ab}$. **11.** $\dfrac{b}{b - a}$. **13.** $2a$. **15.** $\dfrac{c - d}{a + b}$.

17. $\dfrac{c + d}{2}$. **19.** $\frac{1}{6}$. **21.** $2b$. **23.** $r = \dfrac{S - a}{S - l}$.

Exercise 43. Page 118

1. $9x^2 - 25y^2$. **3.** $4x^2 + 12x + 9$. **5.** $y^4 - 6wy^2 + 9w^2$.
7. $a^3 - 64$. **9.** $(y + 5z)(y - 5z)$. **11.** $(z - 4y)^2$.

13. $(a - 3b)(a^2 + 3ab + 9b^2)$. **15.** $(3y + 2z^2)^2$.
17. $(z + 7)(z - 3)$. **19.** $(4x + 1)(2 - 3x)$. **21.** $5(z - 3w)^2$.
23. $2(a + 2)(a + 1)(a - 1)$. **25.** $(x - a - 3b)(x + a + 3b)$.
27. $\dfrac{3y + 5x}{2y - 3x}$. **29.** $\dfrac{3bx + b^2}{x^2}$. **31.** $\dfrac{2a - b - 5ab - 5b^2}{a^2 - b^2}$.
33. $\dfrac{3b^2 + 3bc + b - 3c}{b^2 - c^2}$. **35.** $-\frac{1}{6}$. **37.** -2. **39.** $-\dfrac{ab}{a + b}$.

Exercise 44. Page 120

13. $(-5, -1)$; area $= 40$ sq. units. **15.** 9 sq. units. **17.** 10 sq. units.
23. All abscissas are 2. **25.** 4 units.

Exercise 45. Page 124

1. (*a*) 8 and 4; (*b*) $-\frac{1}{2}$, $-\frac{3}{2}$, and 0.
15. (*b*) Equals 0 if $x = 4.4$ or 1.6; equals 10 if $x = 6.5$ or $-.5$.

Exercise 46. Page 127

1. 7. **3.** -1. **5.** $\frac{5}{2}$. **7.** -33. **9.** $\frac{1}{4}$.
11. $4c - 12c^2$. **13.** 9; $b^2 - b + 3$; $c^4 - c^2 + 3$; $x^2 - 5x + 9$.
15. 4; 4; $\frac{125}{8}$; $(x + 2y)/(x - y)$. **17.** -5; 27; $c^2 + 6bc$.

Exercise 47. Page 130

19. $x = 5$; $x = -4$. **21.** Cuts x-axis at $(5, 0)$; y-axis at $(0, 3)$.
23. Cuts x-axis at $(-\frac{5}{3}, 0)$; y-axis at $(0, \frac{5}{2})$. **25.** $y = \frac{3}{5}x - \frac{11}{5}$.

Exercise 48. Page 132

Note. In this answer book, in any solution of a system of equations, the values of the unknowns will be arranged in their alphabetical order.

1. $(-\frac{2}{3}, -1\frac{2}{3})$. **3.** $(2, 5)$. **5.** $(-2\frac{2}{3}, 3)$. **7.** $(-2\frac{1}{3}, -\frac{5}{6})$.
9. $(1\frac{2}{3}, -\frac{2}{3})$. **11.** No solution; parallel lines.
13. No solution; parallel lines. **15.** Infinitely many solutions.

Exercise 49. Page 134

1. $(3, 2)$. **3.** $(-1, -3)$. **5.** $(0, -4)$. **7.** $(2, 2)$.
9. $(-\frac{5}{2}, \frac{3}{2})$. **11.** $(\frac{1}{2}, \frac{1}{3})$. **13.** $(2, \frac{3}{5})$. **15.** $(5, 2)$.

Exercise 50. Page 135

1. $(5, 2)$. **3.** $(7, \frac{3}{2})$. **5.** $(0, 0)$. **7.** $(0, 0)$.
9. $(3, -2)$. **17.** $(-\frac{37}{21}, -\frac{19}{7})$. **19.** $(.42, .19)$. **21.** $(-.35, .27)$.
23. $(2, 3)$. **25.** $(5, -3)$. **27.** $(3, 2)$. **29.** $(5, -3)$.

Exercise 51. Page 136

1. $(-1, -\frac{1}{4})$. **3.** $(2, 5)$. **5.** $(\frac{1}{2}, -\frac{3}{2})$.
7. $\left(\dfrac{2}{a}, \dfrac{b}{2}\right)$. **9.** $\left(\dfrac{a}{2b}, \dfrac{b}{a}\right)$. **11.** $\left(\dfrac{3h + k}{9h + 2k}, -\dfrac{hk}{9h + 2k}\right)$.
13. $(2b, -3a)$. **15.** $(a + b, b - a)$. **17.** $\left(m - n, \dfrac{2m - 2n}{}\right)$.

Exercise 52. Page 138

1. $(1, 2, -2)$. **3.** $(-\frac{1}{2}, -\frac{2}{3}, \frac{4}{3})$. **5.** $(\frac{8}{5}, -\frac{2}{5}, \frac{8}{5})$.
7. $(\frac{12}{13}, -\frac{1}{13}, \frac{7}{13})$. **9.** $(-2, 3, 3)$. **11.** $(\frac{1}{2}, \frac{1}{3}, -\frac{1}{2})$. **13.** $(-1, -1, 3, 2)$

Exercise 53. Page 140

1. 30°; 120°. **3.** $41\frac{1}{2}$°; $48\frac{1}{2}$°.
5. $5\frac{1}{3}$ gal. of 20%; $2\frac{2}{3}$ gal. of 50%. **7.** 11′ by 3′.
9. 1st, 3 lb.; 2d, 6 lb. **11.** 13, or 26, or 39.
13. 40 lb. silver; 80 lb. lead. **15.** \$3000 at 3%; \$2500 at 4%; \$4500 at 6%.
17. 465. **19.** $y = -4x - 11$. **21.** $y = -\frac{3}{4}x + 2$.
23. $y = -2$. **25.** Land, 90 mi; water, 48 mi.

Exercise 54. Page 143

1. 32. **3.** -243. **5.** $\frac{9}{16}$. **7.** Minus. **9.** x^{4+c}.
11. x^{15}. **13.** $32a^{10}$. **15.** $625x^8y^4$. **17.** $-8x^6$. **19.** $16a^{12}$.
21. a^{2k}. **23.** d^{2hk}. **25.** $c^{nk}d^{3k}$. **27.** $.09c^2d^6$. **29.** $\frac{1}{z^5}$.
31. $\frac{c^3}{d^3}$. **33.** $\frac{9y^2}{4}$. **35.** $\frac{a^2}{b^2}$. **37.** $\frac{b^4}{a^2}$.
39. $\frac{64a^6}{27x^3}$. **41.** $\frac{c^{3k}}{d^{2k}}$. **43.** $\frac{z^{3n}}{a^{3h}}$. **45.** $-\frac{a^3b^6}{27}$.
47. $\frac{c^{nx}d^{ny}}{a^{2n}}$. **49.** $\frac{9y^2z^4}{64x^2}$. **51.** $-\frac{50}{w^4x^2y^2z}$. **53.** $\frac{w^5y^2z^3}{200x^5}$.
55. (*a*) 16; -16: (*b*) n odd.

Exercise 55. Page 147

1. ± 8. **3.** ± 9. **5.** $\pm \frac{1}{3}$. **7.** $\pm .1$. **9.** 12.
11. $\frac{1}{6}$. **13.** $\frac{1}{7}$. **15.** -3. **17.** 5. **19.** -6.
21. ± 3. **23.** ± 5. **25.** $\pm \frac{1}{2}$. **27.** d. **29.** 3.
31. 3. **33.** 57. **35.** $4xy^3$. **37.** 6. **39.** -2.
41. 2. **43.** 4. **45.** 2. **47.** -1. **49.** 6. **51.** 20.
53. 20. **55.** $\frac{1}{2}$. **57.** $-\frac{1}{3}$. **59.** .1. **61.** .1. **63.** .2.

Exercise 56. Page 150

1. b. **3.** a. **5.** x^2. **7.** z^4. **9.** y^3.
11. x^2. **13.** x^3. **15.** $2y$. **17.** $2y$. **19.** $\frac{3}{4}$.
21. $\frac{2}{3}$. **23.** $\frac{3}{10}$. **25.** $\frac{3}{2}$. **27.** $3x^2$. **29.** $-2x$.
31. x^2y^3. **33.** $2a^2$. **35.** $-.1$. **37.** $2xy^2$. **39.** $-2z^2$.
41. $-xz^3$. **43.** $.2x^5$. **45.** $.5x$. **47.** $\frac{2x}{3y^2}$. **49.** $\frac{3x^2}{2y^3}$.

Exercise 57. Page 153

1. 3. **3.** 2. **5.** $\frac{1}{4}$. **7.** $\frac{1}{35}$. **9.** 8.
11. $\frac{1}{9}$. **13.** $\frac{1}{81}$. **15.** $\frac{1}{2}$. **17.** $\frac{1}{3}$. **19.** 1.

21. $\frac{1}{3}$. **23.** 3. **25.** .6. **27.** $\frac{1}{9}$. **29.** $\frac{1}{25}$.
31. $-\frac{1}{32}$. **33.** -1. **35.** $-\frac{1}{2}$. **37.** -5. **39.** 10.
41. 125. **43.** 216. **45.** 625. **47.** $\frac{1}{9}$. **49.** 16.
51. $\frac{1}{b^4}$. **53.** $\frac{y}{x^2}$. **55.** $\frac{c^3}{d^3}$. **57.** $\frac{3}{h^4}$ **59.** $\frac{4y}{x^3}$.
61. $\frac{2a}{y^5}$. **63.** $\frac{a}{4x^3}$. **65.** b^4. **67.** a^3c. **69.** $\frac{1}{5a^3}$.
71. $\frac{b^3}{a^2c^2d^4}$. **73.** $\frac{a^3c}{6d^3}$. **75.** $\frac{4b^7}{3a^6}$. **77.** $\frac{125a^3b^2}{9}$. **79.** $\frac{18z^2}{a^2y^3}$.
81. z^{-5}. **83.** $5y^{-4}$. **85.** $5y^{-4}z^2$. **87.** $4(3^{-1}a^2x^{-1}y^{-3})$.
89. $8x^{\frac{1}{2}}y^{-\frac{1}{2}}z^{-1}$. **91.** $3(1.04)^{-16}$. **93.** $c(x-5y)^{-1}$. **95.** $\sqrt[5]{z}$.
97. $\sqrt[3]{b^4}$. **99.** $5\sqrt[3]{c}$. **101.** $b\sqrt[3]{x^2}$. **103.** $b^{\frac{6}{7}}$. **105.** $\sqrt[5]{6c}$.
107. $\sqrt{2xy}$. **109.** $\sqrt[3]{49a^4}$. **111.** $a^{\frac{2}{3}}$. **113.** $5^{\frac{1}{4}}c^{\frac{3}{4}}$. **115.** $(a^2-3b)^{\frac{1}{2}}$.
117. $(c-3d)^{\frac{2}{5}}$. **119.** $(a^3-b^3)^{\frac{1}{3}}$. **121.** $(4-x^2)^{\frac{1}{2}}$.

Exercise 58. Page 155

1. $x^{\frac{2}{4}}$. **3.** x^4. **5.** a^6. **7.** 16. **9.** $8x^3$.
11. $\frac{125}{a^6}$. **13.** 25. **15.** $a^{\frac{1}{6}}$. **17.** $\frac{1}{x^3}$. **19.** $\frac{1}{b^6}$.
21. $\frac{8}{x^6}$. **23.** $\frac{a^6}{x^9}$. **25.** $\frac{9y^6}{x^2}$. **27.** $\frac{x^2y^6}{36}$.
29. 8. **31.** 625. **33.** $\frac{x^4}{16}$. **35.** $\frac{9x^4}{a^2}$.
37. $x^{\frac{3}{2}}$. **39.** $\frac{1}{a^{\frac{4}{3}}}$. **41.** $\frac{1}{x^{\frac{1}{2}}}$. **43.** $\frac{1}{a^{\frac{5}{3}}x^4}$.
45. $\frac{a^{\frac{1}{2}}}{b^{\frac{3}{2}}}$. **47.** $\frac{a^{\frac{1}{12}}}{2y^{\frac{1}{4}}}$. **49.** ab^3. **51.** $\frac{a^5}{y}$.
53. $\frac{8a^{\frac{15}{2}}x^3}{27}$. **55.** $\frac{a^{\frac{8}{3}}b^2}{81x}$. **57.** $\frac{3}{2xy^2}$. **59.** $9m^4$.
61. $25x^4$. **63.** $\frac{a+b}{ab}$. **65.** $\frac{b^3-a^2}{a^3b^3}$. **67.** $\frac{b}{1+ab}$.
69. $\frac{1}{y^2+c^2}$. **71.** $\frac{ab}{b-a}$. **73.** $\frac{a^2b^2}{b^2+ab+a^2}$. **75.** $\frac{1}{3ab}$.
77. $\frac{ab^2}{a+b^2}$. **79.** $x^{-2}-y^{-2}$. **81.** $16x^2-y^{\frac{2}{3}}$. **83.** $x^{\frac{1}{3}}-y^{\frac{1}{2}}$.
85. $15a+14a^{\frac{1}{2}}-8$. **87.** $a^{-4}+2a^{-2}b+b^2$.
89. $a^{\frac{2}{3}}+2a^{\frac{1}{3}}b^{\frac{2}{3}}+b^{\frac{4}{3}}$. **91.** $a^{-2}+2a^{-1}y^2+y^4$.
93. $a^{-3}+3a^{-2}b+3a^{-1}b^2+b^3$. **95.** $27-27y^{-1}+9y^{-2}-y^{-3}$.
97. $6-x^{-2}-15x^{-4}$. **99.** $3^nx^{\frac{n}{4}}$. **101.** $4^ma^{-km}b^{mn}$.
103. $\frac{8x^{6k}}{343a^{9n}}$. **105.** $a^3-b^{\frac{3}{2}}$. **107.** $(x-y^{-1})(x+y^{-1})$.
109. $(3x^{-1}-b^{-2})(3x^{-1}+b^{-2})$. **111.** $(2x^{\frac{1}{3}}-y^{\frac{1}{5}})(2x^{\frac{1}{3}}+y^{\frac{1}{5}})$.

113. $(3x^{\frac{1}{3}} - 5y^{\frac{1}{2}})(3x^{\frac{1}{3}} + 5y^{\frac{1}{2}})$. **115.** $(2a^{\frac{1}{2}} - 3b^{\frac{1}{3}})(2a^{\frac{1}{2}} + 3b^{\frac{1}{3}})$.
117. $(z - 3x^{-1})^2$. **119.** $(3a^{-1} - b^{-2})^2$. **121.** $(6x^{\frac{2}{3}} + y^{\frac{2}{3}})^2$.
123. $(3x^{-1} - 2y)(x^{-1} + y)$. **125.** $(2a^{\frac{1}{3}} + 3b^{\frac{1}{3}})(4a^{\frac{2}{3}} - 6a^{\frac{1}{3}}b^{\frac{1}{3}} + 9b^{\frac{2}{3}})$.
127. $(6 - x^{\frac{1}{3}})(36 + 6x^{\frac{1}{3}} + x^{\frac{2}{3}})$. **129.** $3x^{-1} + y^{-1}$.

Exercise 59. Page 158

1. $3\sqrt{2}$; 4.242. **3.** $2\sqrt{5}$; 4.472. **5.** $10\sqrt{2}$; 14.14.
7. $3\sqrt{3}$; 5.196. **9.** $6\sqrt{2}$; 8.484. **11.** $.3\sqrt{5}$; .6708.
13. $2\sqrt[3]{2}$; 2.520. **15.** $3\sqrt[3]{4}$; 4.761. **17.** $-\sqrt[3]{5}$; -1.710.
19. $-3\sqrt[3]{2}$; -3.780. **21.** $x^3\sqrt{x}$. **23.** $y\sqrt[3]{y^2}$.
25. $x^2\sqrt[4]{x^2}$. **27.** $3a^2$. **29.** $2a\sqrt[3]{a}$. **31.** $2z\sqrt[4]{z^3}$.
33. $2ay^2\sqrt{3ay}$. **35.** $ay\sqrt[3]{12y^2}$. **37.** $3y\sqrt[3]{2y^2}$. **39.** $2y^2z\sqrt[3]{2y^2z}$.
41. $-ay^2\sqrt[3]{a^2}$. **43.** $xy^2\sqrt[4]{3y^2}$. **45.** $2d\sqrt[4]{c^3d}$. **47.** $-a^2y^2\sqrt[3]{9a^2}$.
49. $.5x^3\sqrt{x}$. **51.** $\dfrac{2x^2}{y^2}\sqrt{x}$. **53.** $\dfrac{x}{2y^2}\sqrt[3]{5x^2}$. **55.** $\dfrac{6x}{5y}\sqrt[3]{x^2}$.
57. $-\dfrac{2}{ab^2}\sqrt[3]{2}$. **59.** $3\sqrt{1 + y^2}$. **61.** $a\sqrt{1 + 5b}$. **63.** $2a\sqrt[3]{2 - az^3}$.
65. x^h. **67.** $2x^n\sqrt[3]{2}$. **69.** $\dfrac{(a + b)\sqrt{d}}{ab}$. **71.** $8\sqrt{2}$. **73.** $11\sqrt{2}$.
75. $3\sqrt{3}$. **77.** $(a - 5b)\sqrt{2}$. **79.** $(3 - x)\sqrt[3]{3x}$. **81.** $(2x - 2)\sqrt[4]{3y}$.

Exercise 60. Page 160

1. $\sqrt{15}$; 3.873. **3.** $5\sqrt{2}$; 7.070. **5.** $6\sqrt{2}$; 8.484.
7. 40. **9.** $18\sqrt{2}$; 25.45. **11.** $5\sqrt{42}$; 32.40.
13. 54. **15.** $9\sqrt[3]{60}$; 35.24. **17.** $-2\sqrt[3]{3}$; -2.884.
19. $\sqrt{7}$; 2.646. **21.** $\sqrt[3]{9}$; 2.080. **23.** $\sqrt{5}$. **25.** $\dfrac{x}{2}$.
27. $\dfrac{\sqrt[3]{3}}{a}$. **29.** $3x^2\sqrt{5}$. **31.** $3x\sqrt{2x}$. **33.** $3ab\sqrt[3]{2a^2b}$.
35. $375a^2$. **37.** $54x$. **39.** $b^2x^2 + b^2$. **41.** $-9 + 7\sqrt{5}$.
43. 1. **45.** $18 + 13\sqrt{6}$. **47.** 5. **49.** $2\sqrt{6} + \sqrt{10} + 8\sqrt{3} + 4\sqrt{5}$.
51. $27 + 10\sqrt{2}$. **53.** $14 - 4\sqrt{6}$. **55.** $a - 9by$. **57.** $6x - 15y + \sqrt{xy}$.
59. $a + b^2x + 2b\sqrt{ax}$. **61.** $-xyz\sqrt[5]{z^4}$. **63.** $\sqrt{18a}$.
65. $\sqrt{a^2bx}$. **67.** $\sqrt[3]{27b^2}$. **69.** $\sqrt[4]{48b}$.

Exercise 61. Page 162

1. $\frac{1}{2}\sqrt{2}$; .707. **3.** $\frac{1}{5}\sqrt{10}$; .632. **5.** $\frac{1}{4}\sqrt{10}$; .7905. **7.** $\frac{1}{2}\sqrt[3]{2}$; .630.
9. $\frac{1}{4}\sqrt[3]{20}$; .6785. **11.** $-\frac{1}{10}\sqrt[3]{7}$; $-.2646$. **13.** $\frac{1}{100}\sqrt{30}$; .05477.
15. $-\frac{1}{10}\sqrt[3]{30}$; $-.3107$. **17.** $\frac{1}{50}\sqrt{30}$; .1095.

19. $\frac{1}{3}\sqrt{3x}$. 21. $\frac{\sqrt{2a}}{a}$. 23. $\frac{\sqrt{3}}{3x}$. 25. $\frac{1}{2}\sqrt[3]{2c}$.

27. $\frac{\sqrt[3]{4ab^2}}{2b}$. 29. $\frac{\sqrt{10a}}{2b}$. 31. $\frac{\sqrt{5z}}{5wz}$. 33. $\frac{\sqrt[3]{3a^2b}}{3a^2}$.

35. $\frac{\sqrt[3]{cd}}{2c}$. 37. $\frac{2n\sqrt[3]{9x}}{3x^2}$. 39. $\frac{1}{3}\sqrt[4]{3cx^2}$. 41. $\frac{\sqrt[4]{3a^2xy}}{ax}$.

43. $-\frac{\sqrt[3]{25b}}{5ab^3}$. 45. $\frac{2\sqrt{ah+hx}}{a+x}$. 47. $\frac{\sqrt{x}}{x^3}$. 49. $\frac{\sqrt[3]{x}}{x}$.

51. $\frac{\sqrt{5x}}{5x^2}$. 53. $\frac{\sqrt{5b(5a^2-9b)}}{5ab}$. 55. $\frac{\sqrt{7x(112x^3+3)}}{7x^2}$. 57. $\frac{\sqrt{c(d^2+c^3)}}{c^3d}$.

59. $\frac{\sqrt[3]{9y(3ay^2+1)}}{3y}$. 61. $4\sqrt{5}$. 63. 0. 65. $\frac{\sqrt[3]{a^2}}{a}+3a\sqrt[3]{a}$.

Exercise 62. Page 163

1. $\frac{1}{3}\sqrt{3}$; .577. 3. $\frac{6}{5}\sqrt{5}$; 2.683. 5. $\frac{2}{3}\sqrt{3}$; 1.155.

7. $\frac{1}{3}\sqrt{15}$; 1.291. 9. $\frac{2}{5}\sqrt{15}$; 1.549. 11. $\frac{3}{2}\sqrt[3]{2}$; 1.890.

13. $\frac{9-5\sqrt{3}}{6}$; .057. 15. $3-2\sqrt{2}$; .172. 17. $\frac{7-3\sqrt{6}}{5}$; $-$ .0694.

19. $\frac{1-\sqrt{15}}{7}$; $-$ .410. 21. $\frac{17+4\sqrt{10}}{43}$; .689. 23. $\frac{-19+3\sqrt{42}}{17}$; .026.

25. $\frac{8-5\sqrt{2}}{2}$; .465. 27. $-\frac{1}{6}\sqrt[3]{18}$; $-$ .437. 29. $-\frac{1}{10}\sqrt[3]{100}$; $-$ .4642.

31. $\frac{1}{5}\sqrt[3]{5}$; .342. 33. $\frac{\sqrt{3}}{3x}$. 35. $-\frac{\sqrt[3]{2t^2z^2}}{2t}$.

37. $-\frac{\sqrt[3]{4ab^2}}{2ab}$. 39. $\frac{\sqrt[3]{4b^2c^2}}{2bc}$. 41. $\frac{\sqrt[6]{27xy^3}}{3y}$.

43. $\frac{2\sqrt{x^2-4}-3\sqrt{x-2}}{4x-1}$. 45. $\frac{3\sqrt[3]{4}+4\sqrt[3]{2}}{2}$; 4.900.

47. $\frac{2\sqrt{5}+4\sqrt{3}-\sqrt{6}+3\sqrt{10}}{14}$; 1.317. 49. $\frac{\sqrt[n]{3x^{n-2}}}{x}$. 51. $\frac{b\sqrt[k]{a^{k-2}x}}{ax^2}$.

Exercise 63. Page 166

1. $\sqrt[6]{a^2b^3}$. 3. $5\sqrt[4]{a^3}$. 5. $2\sqrt[12]{a^3b^8}$. 7. $b\sqrt[6]{a^4b^3}$.

9. $\sqrt[4]{x^2y^3}$. 11. $\sqrt[3]{y}$. 13. $\sqrt{x}$. 15. $\sqrt[3]{y^2}$.

17. $\sqrt[3]{z}$. 19. $\sqrt{3}$. 21. $\sqrt{3}$. 23. $\sqrt[4]{6}$.

25. $\sqrt{2}$. 27. $\sqrt{3a}$. 29. $\sqrt{2a}$. 31. $b\sqrt{b}$.

33. $x^2\sqrt{x}$. 35. $\sqrt[3]{x^2}$. 37. 9. 39. $5\sqrt[3]{5}$.

41. $25\sqrt{5}$. 43. $9a^2$. 45. $3a^7\sqrt{3a}$. 47. $48\sqrt[3]{3}$.

49. $\sqrt[6]{x}$. 51. $\sqrt[6]{y}$. 53. $\sqrt[4]{3}$. 55. $\sqrt[6]{a}$.

57. $\sqrt[4]{y}$. 59. $\sqrt[4]{a}$. 61. $\sqrt[6]{a^5}$. 63. $\sqrt[12]{y^7}$.

65. $\sqrt[4]{27}$. **67.** $a\sqrt[6]{a}$. **69.** $\dfrac{\sqrt[6]{a^5}}{a}$. **71.** $\dfrac{\sqrt[4]{b^3}}{b}$.

73. $\sqrt[12]{5}$. **75.** 1. **77.** $\frac{1}{2}\sqrt[6]{54}$. **79.** $\dfrac{\sqrt[4]{c^3d}}{c}$.

81. $\frac{1}{5}\sqrt{10}$. **83.** $-\dfrac{x\sqrt[3]{4x}}{2}$. **85.** $\dfrac{\sqrt[4]{9x}}{3x}$. **87.** $\dfrac{2\sqrt[3]{2y^2}}{y^2}$.

89. $\sqrt[4]{2a}$. **91.** $ax^2\sqrt[4]{ax^2}$. **93.** $b\sqrt[6]{9b^3}$. **95.** $3\sqrt[12]{3}$.

97. $\dfrac{\sqrt{x^4+y^2}}{x^2y}$. **99.** $\dfrac{a\sqrt[3]{4a^2b^2}}{2b}$. **101.** $\dfrac{y\sqrt{6xy}}{2b}$. **103.** $a^2x^4\sqrt[3]{ax^2}$.

105. $\sqrt[12]{a}$. **107.** $\sqrt[3]{3y}$. **109.** $2a^5\sqrt[3]{4}$. **111.** $c^3\sqrt[5]{64}$.

113. $2\sqrt[3]{2}$. **115.** $\dfrac{(2+a)\sqrt{a^2-1}}{a(1+a)}$. **117.** $\dfrac{(2b+a)\sqrt{ab}}{b}$.

119. $2(\sqrt{3}+\sqrt{2})$. **121.** $\dfrac{3a+\sqrt{3a}+\sqrt{3a^2+3ab}+\sqrt{a+b}}{b-2a}$.

123. $2x\sqrt[4]{3y}+(1-a)\sqrt[4]{2y}$.

Exercise 64. Page 168

1. $\frac{1}{216}$. **3.** 1. **5.** $3\sqrt{3}$; 5.196. **7.** 125.

9. $\frac{1}{4}$. **11.** $\frac{3}{5}\sqrt{5}$; 1.342. **13.** 239. **15.** $\frac{1}{4}\sqrt[3]{12}$; .572.

17. $\frac{1}{8}\sqrt{6}$; .306. **19.** $\sqrt[3]{2}$; 1.260. **21.** $3\sqrt{2}$; 4.242.

23. $\frac{1}{25}\sqrt{10}$; .1265. **25.** $-\frac{1}{10}\sqrt[3]{100}$; $-$ .2154. **27.** $\dfrac{-1+\sqrt{6}}{5}$; .290.

29. $3a^{-2}b^2x^{-4}$. **31.** $2^{-1}a^{-\frac{1}{2}}b^{-1}$. **33.** $w^{\frac{4}{3}}$.

35. $3z^{\frac{1}{2}}$. **37.** $2z^{\frac{2}{3}}$. **39.** $3^{\frac{1}{2}}c^{\frac{1}{2}}$. **41.** $3b^3$.

43. $\dfrac{8}{x^3}$. **45.** $64x^{\frac{3}{2}}$. **47.** $9b^6$. **49.** $\dfrac{4x^2}{9y^6}$.

51. $\dfrac{\sqrt[6]{3a^5x^5}}{a}$. **53.** $\dfrac{3x}{1+2xy}$. **55.** 4; $-$ 3.

57. $2y\sqrt[5]{2xy^2}$. **59.** $-2x^2\sqrt[3]{4x^2}$. **61.** $\frac{1}{4}\sqrt[3]{6}$.

63. $\frac{1}{2}\sqrt[6]{432}$. **65.** $x^2\sqrt[6]{16x^3}$. **67.** $5x\sqrt[3]{5x}$.

69. $\sqrt[6]{x}$. **71.** $3\sqrt[6]{x}$. **73.** $b\sqrt{b}$. **75.** $\sqrt[3]{6}$. **77.** $\dfrac{\sqrt{6y}}{y}$.

79. $y\sqrt[6]{x^3y}$. **81.** $\dfrac{3z^2\sqrt{2x}}{2x^2}$. **83.** $\dfrac{\sqrt[6]{1125}}{5}$. **85.** $\dfrac{(31-18a)\sqrt{3}}{3}$

87. $-\dfrac{.5z\sqrt[3]{xz}}{x}$. **89.** $\dfrac{\sqrt[6]{(2x-3y)(2x+3y)^4}}{2x+3y}$. **91.** $\dfrac{a+\sqrt{a^2-b^2}}{b}$.

93. $(a-z)\sqrt{3x}$. **95.** $\dfrac{\sqrt[6]{(a+b)(a-b)^4}}{a-b}$. **97.** $\dfrac{\sqrt[6]{b^3(a^2-b)}}{ab}$.

Exercise 65. Page 172

1. $3i$. **3.** $5i$. **5.** $8i$. **7.** $5i\sqrt{2}$. **9.** $i\sqrt{7}$.
11. $\frac{1}{2}i$. **13.** $\frac{1}{7}i$. **15.** $\frac{1}{5}i$. **17.** $.3i$. **19.** $\frac{4}{5}i$.
21. $.6i$. **23.** $\frac{1}{5}i\sqrt{5}$. **25.** $\frac{2}{11}i\sqrt{11}$. **27.** $2bi$. **29.** abi.
31. $2ix\sqrt{2}$. **33.** $5ih\sqrt{3}$. **35.** $8ix^2y^2\sqrt{2y}$. **37.** $3ih^3\sqrt{3h}$.
39. $\frac{1}{5}cdi$. **41.** $\dfrac{3a^2i\sqrt{ac}}{2c^2}$. **43.** $\pm \frac{4}{5}i$. **45.** $\pm 3i\sqrt{7}$.
47. i. **49.** -1. **51.** -1. **53.** 10. **55.** 13.
57. $-29 + 11i$. **59.** $21 - 20i$. **61.** $9 + 40i$. **63.** $8i - 40$.
65. $4 + 19i$. **67.** -15. **69.** $3i\sqrt{2} + 10\sqrt{2}$. **71.** $-20i\sqrt{2} - 17$.
73. $(4i - 19)$; $(-34 - 6i)$; $(-70i - 66)$. **75.** $\dfrac{22 + 7i}{41}$. **77.** $1 + 2i$.

Exercise 66. Page 174

Note. In simplifying radicals in the solution of an equation, it will be assumed that any literal factor of a radicand is positive if this adds to our convenience in the reduction.

1. ± 5. **3.** $\pm 3i$. **5.** $\pm \frac{5}{3}i$. **7.** $\pm \frac{1}{5}\sqrt{35}$; ± 1.183.
9. $\pm \frac{1}{2}\sqrt{c}$. **11.** $\pm \dfrac{\sqrt{3ah}}{3a}$. **13.** $\pm \frac{1}{4}\sqrt{11}$; $\pm .829$.
15. $\pm \frac{1}{3}\sqrt{30}$; ± 1.826. **17.** $\pm \frac{1}{2}\sqrt{2}$; $\pm .707$. **19.** $\pm \frac{4}{3}i\sqrt{2}$.
21. $\pm \frac{1}{2}i\sqrt{10}$. **23.** $\pm \dfrac{\sqrt{2c(d-a)}}{c}$. **25.** $\pm \dfrac{5\sqrt{a(1-b)}}{2(1-b)}$.
27. $\pm \dfrac{5\sqrt{1+a}}{2(1+a)}$. **29.** $v = \pm \dfrac{\sqrt{frm}}{m}$. **31.** $r = \pm \dfrac{\sqrt{\pi A}}{\pi}$.
33. $\pm \frac{1}{2}\sqrt{2}$; $\pm .707$. **35.** ± 14. **37.** $\pm \frac{1}{2}\sqrt{13}$; ± 1.803.

Exercise 67. Page 176

1. 5; -2. **3.** -4; 3. **5.** 0; $\frac{3}{2}$. **7.** 0; $\frac{7}{3}$.
9. 0; $\frac{2}{5}$. **11.** $\pm \frac{5}{2}$. **13.** -3; $\frac{1}{2}$. **15.** $\frac{1}{2}$; $\frac{3}{4}$.
17. $\frac{2}{5}$; $\frac{2}{5}$. **19.** $-\frac{1}{2}$; $-\frac{1}{2}$. **21.** -2; $-\frac{3}{2}$. **23.** $\frac{3}{5}$; -4.
25. $-\frac{5}{4}$; $-\frac{5}{4}$. **27.** $\frac{5}{4}$; -3. **29.** $\frac{5}{7}$; -2. **31.** $-\frac{4}{5}$; 1.
33. $-\frac{2}{3}$; $\frac{3}{2}$. **35.** 0; $\dfrac{3d}{2a}$. **37.** $-3b$; $-2b$. **39.** $-b$; $\frac{4}{3}b$.
41. $-\dfrac{1}{3b}$; $\dfrac{3}{2b}$. **43.** $-\dfrac{2b}{3a}$; $-\dfrac{2b}{3a}$. **45.** $-\frac{5}{3}$; $-\frac{5}{3}$. **47.** 2; $-\frac{16}{5}$.
49. $\frac{1}{6}$. **51.** -3; $-\frac{1}{4}$. **53.** 0; $\frac{3}{2}$; $-\frac{5}{3}$. **55.** -3; $\frac{4}{3}$.

Exercise 68. Page 180

1. 16; $(x - 4)^2$. **3.** c^2; $(x - c)^2$. **5.** $\frac{4}{9}$; $(x - \frac{2}{3})^2$.
7. $\frac{49}{36}$; $(x + \frac{7}{6})^2$. **9.** -7; 1. **11.** 3; -7. **13.** -2; -2.

15. $\dfrac{-2 \pm \sqrt{14}}{2}$: -2.871; $.871$. **17.** $(2+i)$; $(2-i)$.

19. $\dfrac{2 \pm \sqrt{3}}{3}$: 1.244; $.089$. **21.** $\dfrac{3 \pm 2i}{2}$.

23. $\dfrac{-4 \pm \sqrt{19}}{3}$: -2.786; $.120$. **25.** $\frac{7}{3}$; -1.

29. $5a$; $-3a$. **31.** b; $-\frac{3}{2}b$. **33.** $\dfrac{-a \pm \sqrt{a^2+3b}}{3}$.

35. $\dfrac{-2 \pm \sqrt{4+ac}}{a}$. **37.** $\dfrac{-K \pm \sqrt{K^2-4HP}}{2H}$.

Exercise 69. Page 182

1. $\frac{1}{2}$; $-\frac{2}{3}$. **3.** $\frac{3}{2}$; $-\frac{1}{3}$. **5.** $1 \pm 3i$. **7.** $\frac{3}{2}$; $\frac{3}{2}$.

9. $\dfrac{2 \pm \sqrt{3}}{2}$: $.134$; 1.866. **11.** $\pm \frac{7}{6}$. **13.** $\dfrac{-1 \pm \sqrt{2}}{3}$: $-.805$; $.138$.

15. $\dfrac{4 \pm \sqrt{10}}{2}$: 3.581; $.419$. **17.** $\dfrac{1 \pm i\sqrt{11}}{4}$. **19.** $\pm \frac{4}{3}i$.

21. $\dfrac{2 \pm 2i\sqrt{5}}{3}$. **23.** $.3$; $.5$. **25.** $\dfrac{1 \pm 2i\sqrt{13}}{2}$.

27. $\dfrac{-2 \pm i\sqrt{5}}{2}$. **29.** $\dfrac{3 \pm 2i}{2}$. **31.** $\frac{2}{5}$; $\frac{2}{5}$.

33. $\dfrac{5 \pm \sqrt{2}}{3}$: 2.138; 1.195. **35.** $\dfrac{-2 \pm 5i}{2}$. **37.** $\frac{3}{2}d$; $-\frac{2}{3}d$.

39. $\dfrac{d \pm \sqrt{d^2-12ac}}{2a}$. **41.** $\dfrac{3k \pm \sqrt{9k^2-120k}}{10k}$. **43.** $-d$; $-2c$.

45. -2; $\dfrac{3}{5k}$. **47.** $\dfrac{2}{3}$, $\dfrac{5}{2h}$. **49.** $\dfrac{2}{1+h}$, $-\dfrac{5}{3}$.

51. $y = x + 2$; $y = \frac{1}{2}(1-x)$.

53. $x = y - 2$; $x = 1 - 2y$.

Exercise 70. Page 184

1. -3; $\frac{1}{2}$. **3.** $\frac{3}{4}$; $-\frac{5}{3}$. **5.** 9; -5. **7.** $-\frac{3}{7}$; $\frac{1}{2}$.

9. $\frac{2}{7}$; $\frac{2}{7}$. **11.** $\frac{1}{5}$; $\frac{1}{5}$. **13.** $\dfrac{2 \pm \sqrt{5}}{5}$: $.847$; $-.047$. **15.** 0; $\frac{7}{6}$.

17. $\pm \frac{3}{11}\sqrt{11}$; $\pm .905$. **19.** $\pm \frac{2}{5}i\sqrt{5}$. **21.** $\frac{3}{4}$; $-\frac{7}{5}$.

23. $6 \pm \sqrt{41}$: 12.403; $-.403$. **25.** 4; $-\frac{5}{2}$. **27.** 6; $\frac{1}{2}$.

29. $-\frac{1}{2}$; $\frac{1}{2}(1-b)$. **31.** $-k$; $-\frac{1}{3}h$. **33.** $\dfrac{2k-h \pm \sqrt{(2k-h)^2+5c}}{c}$.

35. $\dfrac{3c \pm \sqrt{9c^2-4dh}}{2d}$. **37.** 13′ by 17′. **39.** $\frac{4}{17}$; $-\frac{17}{4}$. **41.** 14.928′.

43. 6.48 yd. **45.** 20 m.p.h. **47.** 6 m.p.h.

49. (a) $t = \dfrac{v \pm \sqrt{v^2 - 2gs}}{g}$; (b) $s = 500'$ at $t = 3.46$ sec. and 9.04 sec.; $s = 0'$ at $t = 0$ sec. and $t = 12.5$ sec.

Exercise 71. Page 188

1. Vertex (0, 0); axis $x = 0$; min. = 0.
3. Vertex (0, 0); axis $x = 0$; max. = 0.
5. Vertex (0, 5); axis $x = 0$; min. = 5.
7. Vertex $(-3, -4)$; axis $x = -3$; min. $= -4$.
9. Vertex (1, 5); axis $x = 1$; max. = 5.
11. Vertex $(-2, -5)$; axis $x = -2$; min. $= -5$.
13. Vertex $(\frac{3}{2}, -9)$; axis $x = \frac{3}{2}$; min. $= -9$.
15. Min. $= -13$. **17.** Max. = 8. **19.** At end $2\frac{1}{2}$ sec.
27. 30; 30. **29.** $7\frac{1}{2}''$ by $15''$.

Exercise 72, Page 190

1. $\frac{5}{2}$. **3.** 3; 3. **5.** Roots imag. **7.** 1.6; -4.1. **9.** Roots imag.
11. 2.6; $-.6$. **13.** -3; -1. **15.** $(0, 2\frac{1}{2})$; $(1, 1\frac{1}{2})$; $(3\frac{1}{2}; -1)$.

Exercise 73. Page 192

1. Disc. = 9; real, unequal, and rational.
3. Disc. = 12; real, unequal, and irrational.
5. Disc. = 0; real, equal, and rational.
7. Disc. = 1705; real, unequal, and irrational.
9. Disc. = 0; real, equal, and rational.
11. Disc. $= -16$; imaginary and unequal. **19.** 5.2; -1.2. **21.** -1.2; .2.
23. Disc. = 49; graph is a parabola concave upward, with its axis perpendicular to the x-axis, cutting x-axis in two points, and hence the vertex is below that axis.
25. Disc. $= -59$; graph is a parabola concave downward, with its axis perpendicular to x-axis, which does not meet that axis and hence lies entirely below it.
27. Disc. = 52; etc. **29.** $-2 + 5i$. **31.** $-6i$.

Exercise 74. Page 195

1. -5; -3. **3.** $\frac{3}{4}$; $\frac{3}{2}$. **5.** $-\frac{3}{4}$; $-\frac{7}{4}$. **7.** 0; $-\frac{18}{5}$.
9. $-\frac{7}{9}$; $-\frac{5}{9}$. **11.** $\frac{4}{7}$; $-\frac{5}{7}$. **13.** $-\dfrac{d}{a}$, $-\dfrac{h}{a}$. **15.** $\dfrac{a}{4}$, $-\dfrac{c}{4}$.
17. $-\dfrac{3}{5+a}$, $\dfrac{d}{5+a}$. **19.** $\dfrac{2-c}{1+c}$, $\dfrac{5-d}{1+c}$. **21.** $6x^2 + 7x - 3$.
23. $x^2 + 2x - 7$. **25.** $x^2 + 5x + 6 = 0$. **27.** $3x^2 - 7x + 2 = 0$.
29. $3x^2 - x - 10 = 0$. **31.** $x^2 - 2 = 0$. **33.** $x^2 - 18 = 0$.
35. $9x^2 + 4 = 0$. **37.** $x^2 + 4x - 1 = 0$. **39.** $2x^2 - 2x - 13 = 0$.

41. $x^2 - 6x + 34 = 0$. **43.** $x^2 - 8x + 20 = 0$. **45.** $x^2 - 4x + 24 = 0$.
47. $3x^2 + 4x + 2 = 0$. **49.** $(9x - 8)(3x + 5)$. **51.** $(8x - 15)(3x + 4)$.
53. $27x^2 + 12x - 32 = 0$. **55.** No. (Disc. is not a perfect square.)
57. $(x + 3 + i)(x + 3 - i)$. **59.** $(x - \frac{1}{2} + \frac{3}{2}i)(x - \frac{1}{2} - \frac{3}{2}i)$.

Exercise 75. Page 197

1. $\pm 1;\ \pm 2$. **3.** $\pm 2;\ \pm 2$. **5.** $\pm 2i;\ \pm \frac{1}{2}$. **7.** $\pm 3i;\ \pm \sqrt{2}$.
9. $\pm \frac{2}{3};\ \pm \frac{2}{3}i$. **11.** $\frac{1}{3};\ 1$. **13.** $\pm 2i;\ \pm \frac{1}{3}\sqrt{3}$. **15.** $\pm 5;\ \pm 1$.
17. $-1;\ -\frac{3}{2}$. **19.** $3;\ \pm 2;\ -1$. **21.** $-1;\ \frac{3}{2};\ \frac{1}{4}(1 \pm i\sqrt{11})$.
23. $-\frac{1}{2};\ -\frac{1}{3}$. **25.** $1;\ -4;\ \frac{1}{2}(-3 \pm \sqrt{5})$. **27.** $1;\ -3;\ \frac{1}{2}(3 \pm \sqrt{21})$.
29. $1;\ 3;\ -2;\ -4$. **31.** $2;\ -\sqrt[5]{2}$. **33.** $\frac{2}{3};\ -1;\ \frac{1}{2}(1 \pm i\sqrt{7})$.
35. $\pm 2\sqrt{a};\ \pm \frac{1}{2}\sqrt{6a}$. **37.** $\frac{2}{3};\ \frac{1}{3}(-1 \pm i\sqrt{3})$. **39.** $3;\ \frac{1}{2}(-3 \pm 3i\sqrt{3})$.
41. $\pm \frac{3}{5};\ \pm \frac{3}{5}i$. **43.** $-\frac{3}{5};\ \frac{1}{10}(3 \pm 3i\sqrt{3})$. **45.** $4;\ (-2 \pm 2i\sqrt{3})$.
47. $-1;\ \frac{1}{2}(1 \pm i\sqrt{3})$. **49.** $\frac{1}{2};\ \frac{1}{4}(-1 \pm i\sqrt{3})$. **51.** $\pm 1;\ \pm i$.
53. $\pm 3;\ \pm 3i$. **55.** $\pm 2;\ \pm 2i$. **57.** $\pm \frac{2}{3};\ \pm \frac{2}{3}i$.

Exercise 76. Page 200

1. 7. **3.** No sol. **5.** 12. **7.** -13. **9.** 14. **11.** $\frac{5}{3}\sqrt{2}$.
13. No sol. **15.** 4. **17.** 9. **19.** $4;\ -2$. **21.** $0;\ \frac{3}{2}\sqrt{5}$. **23.** 0.
25. No sol. **27.** $3;\ -1$. **29.** $\frac{1}{4}$. **31.** $4;\ \frac{4}{9}$. **33.** $1;\ \pm 2;\ -3$.
35. a. **37.** $0;\ 4b$. **39.** $\dfrac{gt^2}{\pi^2};\ \dfrac{l\pi^2}{t^2}$. **41.** $\frac{4}{25}$.
43. $1;\ \frac{9}{25}$. **45.** $8;\ -\frac{27}{8}$. **47.** $16;\ \frac{16}{81}$. **49.** 16.
51. $\pm \frac{1}{2}\sqrt{2}$. **53.** 4. **55.** No solution (any principal root is positive).
57. ± 243. **59.** -243. **61.** $-25\frac{1}{3}$.
63. $\sqrt[3]{4}$ ($x^{-\frac{3}{2}} = -8$ has no real solution).

Exercise 77. Page 203

1. $\pm \frac{4}{3}$. **3.** 2. **5.** $0;\ -\frac{12}{3}$.
7. $\frac{11}{13}$ ($k = -1$ is not a solution because it does not give a quadratic equation).
9. $-10;\ -2$. **11.** $\frac{40}{9}$; $k = 0$ does not apply.
13. $-.268;\ -3.732$. **15.** $-\frac{1}{2}$. **17.** $\frac{3}{10}$. **19.** $-\frac{1}{5}$.
21. $\frac{18}{31}$. **23.** $-\frac{9}{32}$. **25.** $\pm 3\sqrt{5}$. **27.** $-\frac{3}{5}$.
29. $-\frac{5}{3}$. **31.** $\pm .816$. **33.** $1;\ -2$.

Exercise 78. Page 206

1. $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$.
3. $x^8 - 8x^7y + 28x^6y^2 - 56x^5y^3 + 70x^4y^4 - 56x^3y^5 + 28x^2y^6 - 8xy^7 + y^8$.
5. $16 + 32a + 24a^2 + 8a^3 + a^4$.
7. $729b^6 - 1458b^5y + 1215b^4y^2 - 540b^3y^3 + 135b^2y^4 - 18by^5 + y^6$.
9. $a^3 + 3a^2b^2 + 3ab^4 + b^6$.
11. $a^{12} - 6a^{10}b^2 + 15a^8b^4 - 20a^6b^6 + 15a^4b^8 - 6a^2b^{10} + b^{12}$.
13. $x^5 - \frac{5}{2}x^4 + \frac{5}{2}x^3 - \frac{5}{4}x^2 + \frac{5}{16}x - \frac{1}{32}$.

15. $x^3 - 6x^{\frac{5}{2}}y^{\frac{1}{2}} + 15x^2y - 20x^{\frac{3}{2}}y^{\frac{3}{2}} + 15xy^2 - 6x^{\frac{1}{2}}y^{\frac{5}{2}} + y^3$.
17. $a^4 - 4a^3y^{-2} + 6a^2y^{-4} - 4ay^{-6} + y^{-8}$.
19. $x^2 - \frac{8x^{\frac{3}{2}}}{a} + \frac{24x}{a^2} - \frac{32x^{\frac{1}{2}}}{a^3} + \frac{16}{a^4}$.
21. $\frac{16}{a^4} - \frac{96b^2}{a^3} + \frac{216b^4}{a^2} - \frac{216b^6}{a} + 81b^8$.
23. $a^{15} + 180a^{14} + 15{,}120a^{13}$.
25. $a^{40} + 20a^{38}b^3 + 190a^{36}b^6$.
27. $1 - 2.2 + 2.31$.
29. $1 - 12\sqrt{2} + 132$.
31. $2^{30}x^{30} - 30 \cdot 2^{29}a^2x^{29} + 435 \cdot 2^{28}a^4x^{28}$.
33. $a^{-26} + 78a^{-25} + 2925a^{-24}$.
35. $x^n - nx^{n-1}y + \frac{n(n-1)}{2}x^{n-2}y^2$.
37. $x^{2m} - mx^{2m-2}y + \frac{m(m-1)}{2}x^{2m-4}y^2$.
39. 720.
41. 39,916,800.
43. 126.

Exercise 79. Page 208

1. $126a^4y^5$.
3. $35x^3y^4$.
5. $-84a^6x^3$.
7. $21a^4x^5$.
9. $-4375x^4y^3$.
11. .00056.
13. $15x^2z^8$.
15. $\frac{n(n-1)\cdots(n-5)}{6!}x^{n-6}y^6$.
17. $672y^2x^{\frac{5}{3}}$.
19. $5670a^4x^8$.
21. $4032x^{\frac{5}{2}}y^8$; $2016x^2y^{10}$.
23. $\frac{945a^3z^{12}}{x^3}$.
25. $70a^4x^4$.
27. $126a^{10}b^4$; $126a^8b^5$.
29. $10{,}000 - 4000a + 600a^2 - 40a^3 + a^4$.
31. 96,059,601.
33. 132,651.
35. 1.127.
37. 1.230.
39. .904.
41. 1.243.
43. .002.
45. 14,776,336.

Exercise 80. Page 211

1. $\frac{6}{5}$.
3. $\frac{3}{4}$.
5. z/a^3.
7. $\frac{54}{13}$.
9. $\frac{1760}{127}$.
11. $\frac{47}{5}$.
13. 30.
15. $\frac{5}{4}$.
17. 10″; 8″.
19. 20; 70.
21. 32″; $21\frac{1}{3}$″.
23. 5500 sq. in.
25. 5.366′.
27. 28.7′.
29. $\pm\sqrt{ab}$.
31. ± 1.
33. ± 25.
35. $\pm 9i$.
37. $\pm\frac{y}{x^2}$.
39. $\pm(y+3)$.
41. -35.
43. $\frac{a^2b^2}{3}$.
45. $\frac{27}{175}$.
47. $\frac{9m^4}{5n}$.

Exercise 81. Page 217

1. $H = \frac{kx}{w^2}$.
3. $Z = \frac{k\sqrt{x}}{y^2}$.
5. $x + 2 = \frac{k}{y+3}$.
7. $V = kr^3$.
9. $W = \frac{k}{d^2}$.
11. $P = k\sqrt{h}$.
13. z is proportional to x^2.
15. u varies directly as x^3 and y.
21. $R = \frac{12y}{5x}$.
23. $H = -\frac{3xy}{\sqrt{z}}$.
25. $\frac{20}{3}$.
27. 784′.
29. 2700 lb.
31. $4\frac{4}{5}$ ft.
33. (a) $71\frac{3}{7}$ lb.; (b) 7 sq. in.
37. $5\frac{1}{3}$ ft. in diameter.
39. 1824 r.p.m.
41. $f_1 : f_2 = 1:16$.
43. 56%.
45. $(10, -6, 4)$.
47. $(6, -2, 4)$ or $(-6, 2, -4)$.
49. $(150, 250, 550, 300)$.

Exercise 82. Page 223

1. 15; 18; 21; 24; 27; 30. **3.** -18; -16; -14; -12; -10; -8.
9. 13. **11.** 14. **13.** 151. **15.** -76. **17.** 10. **19.** $l = 78$; $S = 645$.
21. $l = -72$; $S = -882$. **23.** $l = .78$; $S = 46.86$.
25. $d = 16$; $S = 5460$. **27.** $n = 92$; $S = 18{,}308$.
29. $a = 138$; $S = 2025$. **31.** $n = 21$; $S = 525$.
33. $a = -\frac{13}{4}$; $l = 126\frac{1}{4}$. **35.** $k = -1$.
37. 21. **39.** 136th. **41.** $\frac{45}{31}$.

Exercise 83. Page 226

1. 5; 8; 11; 14. **3.** $8\frac{2}{3}$; $10\frac{1}{3}$; 12; $13\frac{2}{3}$; $15\frac{1}{3}$.
5. 10.5; 6; 1.5; -3; -7.5; -12. **7.** $-\frac{1}{4}$; 1; $\frac{9}{4}$; $\frac{7}{2}$; $\frac{19}{4}$. **9.** 26.
11. -19. **13.** 36,270. **15.** 2223. **17.** 376.
19. 360′. **21.** \$11,650. **23.** \$30,500. **25.** 2565

Exercise 84. Page 230

1. 5; 15; 45; 135. **3.** 4; -8; 16; -32. **7.** $\frac{8}{81}$; $\frac{8}{243}$. **9.** ax^4; ax^6.
11. 1.01; $(1.01)^3$. **13.** 4. **15.** $\frac{8}{5}$. **17.** 729. **19.** $\frac{3}{32}$.
21. $l = 2916$; $S = 4372$. **23.** $l = -1215$; $S = -910$.
25. $l = 192$; $S = 129$. **27.** $l = 384b^7$; $S = \dfrac{3 - 768b^8}{1 - 2b}$.
29. $\frac{1023}{128}$. **31.** $n = 8$; $S = 1275$. **33.** $n = 6$; $S = 111.111$.
35. $a = 25$; $n = 5$. **37.** $n = 11$; $S = \frac{2047}{4}$. **39.** $l = 80$.
41. $\frac{5}{2}$. **43.** (4; 8; 16; 32; 64) or (-4; 8; -16; 32; -64).
45. (12; 36; 108) or (-12; 36; -108).
47. 1; 10; 100; 1000; 10,000; 100,000. **49.** 2. **51.** 10.
53. $\sqrt{xy}$, if $x > 0$; $-\sqrt{xy}$ if $x < 0$. **55.** $\dfrac{(1.05)^{67} - 1}{.05}$.
57. $\dfrac{(1.06)^{30} - (1.06)^4}{.06}$. **59.** $\dfrac{1 - (1.02)^{-15}}{.02}$. **61.** $\dfrac{(1.02)^{10} - 1}{(1.02)^{\frac{1}{2}} - 1}$.
63. $-\frac{3}{16}$; 2. **65.** 8190. **67.** \$102.30.

Exercise 85. Page 233

1. \$95. **3.** 227.8″ approximately. **5.** \$3125.
7. $\frac{1}{2}n(n-1)$. **9.** \$7020. **11.** $\dfrac{x(1 - x^{40})}{1 - x}$.
13. 55.339″ approximately. **15.** At end $2\frac{1}{4}$ yr.
19. 599.59′ approximately. **21.** (*a*) $300(1.06)^6$; (*b*) 5060 units.
23. Approximately 11.1% per year. **27.** $12\frac{1}{2}$%.

Exercise 86. Page 236

1. $\frac{1}{4}$; $\frac{1}{6}$; $\frac{1}{8}$; $\frac{1}{10}$. **3.** $\frac{5}{12}$; $\frac{1}{2}$; $\frac{5}{8}$; $\frac{5}{6}$. **5.** 1; $\frac{5}{7}$; $\frac{5}{9}$; $\frac{5}{11}$; $\frac{5}{13}$.
7. 2. **9.** 16. **11.** $2xy/(x+y)$.

Exercise 87. Page 239

1. 14. **3.** $22\frac{1}{2}$. **5.** $\frac{3}{4}$. **7.** $\frac{100}{101}$. **9.** $\frac{1}{3}$. **11.** $\frac{2}{3}$.
13. $\frac{5}{6}$. **15.** $\frac{7}{33}$. **17.** $\frac{1}{33}$. **19.** $\frac{5}{11}$. **21.** $\frac{19}{90}$. **23.** $\frac{70}{333}$.
25. $\frac{16700}{999}$. **27.** $\frac{518}{3367}$. **29.** 1500″. **31.** 200 sq. in. **33.** 12.

Exercise 88. Page 241

1. 3. **3.** -1. **5.** 64. **7.** 81. **9.** 10. **11.** 1. **13.** $\frac{1}{10}$.
15. $\frac{1}{64}$. **17.** 2. **19.** 2. **21.** 1000. **23.** 6. **25.** 2. **27.** 2.
29. 3. **31.** 4. **33.** $\frac{1}{2}$. **35.** $\frac{1}{3}$. **37.** -1. **39.** -3. **41.** 5.
43. 10. **45.** $\frac{1}{4}$. **47.** 2. **49.** 8. **51.** 100. **53.** 64.

Exercise 89. Page 244

1. .7781. **3.** 1.5314. **5.** 1.4771. **7.** 3.2304.
9. 1.6232. **11.** .3680. **13.** .7533. **15.** $-.3853$.
17. $-.5229$. **19.** -2.1549. **21.** -1.7696. **23.** -1.3768.

Exercise 90. Page 247

1. Ch. = 2; man. = .9356.
3. Ch. = -2; man. = .700. **5.** Ch. = 3; man. = .5473.
7. Ch. = -6; man. = .325. **9.** 9.2562 − 10.
11. 4.4932 − 10. **13.** 5. **15.** -4. **17.** -6.
19. 1.6355. **21.** 7.8949 − 10. **23.** 0.9759. **25.** 4.2504.
27. 8.9345 − 10. **29.** 5.0043. **31.** 5.1959. **33.** 243.
35. 4660. **37.** 1.43. **39.** 74.0. **41.** 302. **43.** .00589.
45. .0960. **47.** .000900. **49.** .264. **51.** .00500.

Exercise 91. Page 251

1. 3.2615. **3.** 2.7261. **5.** 1.5556. **7.** 9.4790 − 10.
9. 9.7503 − 10. **11.** 8.1939 − 10. **13.** 4.9546. **15.** 7.1581 − 10.
17. 6.0910 − 10. **19.** 3.4950. **21.** 1725. **23.** $1.459(10^6)$.
25. 1379. **27.** 39.95. **29.** .0002162. **31.** .4693.
33. $7.695(10^8)$. **35.** 1.030. **37.** .00009738. **39.** .4236.

Exercise 92. Page 253

Note. In some classes, the teacher may desire to teach the use of 5-place logarithms. For the advantage of such classes, in the case of each computation problem in the remainder of this chapter, the result obtained by use of 5-place logarithms is given in black face type beside the result found with 4-place logarithms.

1. 24.91; **24.909.** **3.** .2009; **.20086.** **5.** .006380; **.0063797.**
7. $-.007667$; **− .0076660.** **9.** 51.10; **51.098.** **11.** .1406; **.14061.**
13. 24.56; **24.568.** **15.** .07808; **.078096.** **17.** 5542; **5544.4.**
19. 27.61; **27.609.** **21.** .003467; **.0034669.**

23. $-2.627(10^{-8})$; $\mathbf{-2.6266(10^{-8})}$. **25.** $1.580(10^{-5})$; $\mathbf{1.5802(10^{-5})}$.
27. 38.96; **38.955.** **29.** (a) $4.792(10^5)$; $\mathbf{4.7922(10^5)}$: (b) 8.065; **8.0662.**

Exercise 93. Page 256

1. 5358; **5359.5.** **3.** .4107; **.41082.** **5.** 1.044; **1.0440.**
7. .9500; **.94986.** **9.** 1.315; **1.3158.** **11.** .6030; **.60296.**
13. 28.93; **28.935.** **15.** .1585; **.15849.** **17.** − 1.010; **− 1.0099.**
19. 50.32; **50.324.** **21.** 41.47; **41.470.** **23.** .1266; **.12658.**
25. 2.111; **2.1111.** **27.** 1.041; **1.0412.** **29.** .8630; **.86258.**
31. 50.12; **50.466.** By preliminary use of 7-place table, the results are .5050; **.50504.**
33. 141.9; **141.82.** **35.** 215.1; **215.08.** **37.** .4971; **.49714.**
39. .001352; **.0013525.** **41.** .9388; **.93896.** **43.** .3986; **.39882.**
45. − 1.916; **− 1.9156.** **47.** 21.76; **21.758.** **49.** 134.9; **134.84.**
51. − .136;* **− .1366.*** **53.** 1.118; **1.1177.** **55.** 4.908; **4.9086.**
57. .1730; **.17294.**
59. By 4-place table: (a) $2.219(10^4)$; (b) $3.222(10^{-5})$.
61. .02323; **.023229.** **63.** .0007867; **.0007869.*** **65.** 236.1; **236.13.**

Exercise 94. Page 259

1. 1.341; **1.3410.** **3.** 1.319; **1.3194.** **5.** − 5.195; **− 5.1923.**
7. 18.1;* **18.02.*** **9.** 5.63;* **5.634.*** **11.** 4.317; **4.3176.**
13. 2.303; **2.3026.** **15.** − 14.2;* **− 14.20.***

Exercise 96. Page 261

1. \$4682. **3.** \$4502. **5.** \$1203. **7.** 5.8%. **9.** 18.8 yr. **11.** $16\frac{2}{3}$ yr.

Exercise 98. Page 268

1. (4, .5); (− 2.8, − 2.9). **3.** (5, − 3).
5. (2.1, ± 1.5); (− 2.1, ± 1.5). **7.** (3.2, 3.7). **9.** No real solutions.

Exercise 99. Page 269

1. (3, − 4); (− 4, 3). **3.** (5, − 3); (5, − 3).
5. $\left(\frac{6-2i\sqrt{6}}{3}, \frac{6+i\sqrt{6}}{3}\right)$; $\left(\frac{6+2i\sqrt{6}}{3}, \frac{6-i\sqrt{6}}{3}\right)$.
7. (− 1, − 1); (− 3, 3). **9.** (4, 2); (4, 2). **11.** $(1, \frac{1}{2})$; $(\frac{1}{2}, 2)$.
13. $[\frac{1}{15}(30+2i\sqrt{15}), \frac{1}{15}(45+4i\sqrt{15})]$; $[\frac{1}{15}(30-2i\sqrt{15}), \frac{1}{15}(45-4i\sqrt{15})]$.
15. $(\frac{1}{2}, 1)$; $(-\frac{1}{2}, 3)$.

Exercise 100. Page 270

1. (1.837, ± .790); (− 1.837, ± .790). **3.** $(\frac{4}{3}\sqrt{2}, \pm\frac{4}{3}i)$; $(-\frac{4}{3}\sqrt{2}, \pm\frac{4}{3}i)$.
5. $(\sqrt{5}, \pm 1)$; $(-\sqrt{5}, \pm 1)$. **7.** $(\pm\sqrt{2}, \sqrt{3})$; $(\pm\sqrt{2}, -\sqrt{3})$.
9. $(\frac{1}{2}\sqrt{3}, \pm\frac{1}{2}\sqrt{3})$; $(-\frac{1}{2}\sqrt{3}, \pm\frac{1}{2}\sqrt{3})$. **11.** $(\sqrt{3}, \pm i\sqrt{7})$; $(-\sqrt{3}, \pm i\sqrt{7})$.

* The result is not reliable beyond the last digit given in the answer.

Exercise 101. Page 272

1. $(\sqrt{2}, -\sqrt{2})$; $(-\sqrt{2}, \sqrt{2})$; $(\frac{4}{5}\sqrt{5}, \frac{2}{5}\sqrt{5})$; $(-\frac{4}{5}\sqrt{5}, -\frac{2}{5}\sqrt{5})$.
3. $(\frac{1}{2}, 1)$; $(-\frac{1}{2}, -1)$; $(-\sqrt{2}, \frac{1}{3}\sqrt{2})$; $(\sqrt{2}, -\frac{1}{3}\sqrt{2})$.
5. $(-\sqrt{3}, \frac{2}{3}\sqrt{3})$; $(\sqrt{3}, -\frac{2}{3}\sqrt{3})$; $(-2, 1)$; $(2, -1)$.
7. $(14, -4)$; $(-4, -1)$; $(-14, 4)$; $(4, 1)$.
9. $(\frac{3}{2}, -\frac{7}{2})$; $(\frac{5}{2}, -\frac{1}{2})$, $(-\frac{3}{2}, \frac{7}{2})$; $(-\frac{5}{2}, \frac{1}{2})$.
11. $(-\frac{1}{2}, \frac{3}{2})$; $(\frac{1}{2}, -\frac{3}{2})$; $(-2, 1)$; $(2, -1)$.
13. $(\frac{3}{2}\sqrt{2}, \frac{7}{2}\sqrt{2})$; $(-\frac{3}{2}\sqrt{2}, -\frac{7}{2}\sqrt{2})$; $(2, 5)$; $(-2, -5)$.
15. $(6, 4)$; $(-6, -4)$; $(-\frac{14}{3}i\sqrt{3}, \frac{16}{3}i\sqrt{3})$; $(\frac{14}{3}i\sqrt{3}, -\frac{16}{3}i\sqrt{3})$.

Exercise 102. Page 274

1. $(-\frac{3}{2}, 5)$; $(\frac{5}{2}, -3)$.
3. $(\frac{4}{3}\sqrt{15}, \frac{2}{3}\sqrt{15})$; $(-\frac{4}{3}\sqrt{15}, -\frac{2}{3}\sqrt{15})$.
5. $(-2, 3)$; $(6, -1)$.
7. $(\frac{9}{7}, -\frac{6}{7})$; $(-1, -2)$; $(\frac{3}{7}, \frac{12}{7})$; $(-1, 1)$.
9. $(\frac{1}{2}, 2)$; $(-\frac{1}{2}, -2)$; $(1, 1)$; $(-1, -1)$.
11. $(2, 1)$; $(1, 2)$; $[\frac{1}{2}(-4 + i\sqrt{6}), \frac{1}{2}(-4 - i\sqrt{6})]$;
$[\frac{1}{2}(-4 - i\sqrt{6}), \frac{1}{2}(-4 + i\sqrt{6})]$.
13. $(-5 + i\sqrt{14}, -5 - i\sqrt{14})$; $(-5 - i\sqrt{14}, -5 + i\sqrt{14})$; $(5, 4)$; $(4, 5)$.
15. $(\frac{3}{2}, 2, -1)$; $(\frac{3}{2}, 2, 1)$; $(-\frac{3}{2}, 2, -1)$; $(-\frac{3}{2}, 2, 1)$;
$(\frac{3}{2}, -2, -1)$; $(\frac{3}{2}, -2, 1)$; $(-\frac{3}{2}, -2, -1)$; $(-\frac{3}{2}, -2, 1)$.
17. ± 25.
19. $c = \pm\sqrt{9 + 4m^2}$.
21. $c = \pm\sqrt{a^2 + b^2m^2}$.
23. $(3, 1)$; $(-3, -1)$; $(1, 3)$; $(-1, -3)$.
25. $(\frac{1}{4}, -\frac{1}{7})$; $(\frac{1}{7}, -\frac{1}{4})$.
27. $(\frac{1}{2}, \pm 1)$; $(-\frac{1}{2}, \pm 1)$.

Exercise 103. Page 275

1. $(2, 3)$; $(\frac{3}{2}, 4)$.
3. $(4.1, \pm 1.8)$; $(-4.1, \pm 1.8)$.
5. $(-1.8, -2.1)$; $(2.5, -5.2)$.
7. $(\frac{2}{5}\sqrt{65}, \pm\frac{1}{5}\sqrt{35})$; $(-\frac{2}{5}\sqrt{65}, \pm\frac{1}{5}\sqrt{35})$.
13. $(\frac{2}{3}, -2)$; $(4, -7)$.
15. $(-1, 0)$; $(1, 0)$; $(i\sqrt{2}, -3i\sqrt{2})$; $(-i\sqrt{2}, 3i\sqrt{2})$.
17. $(3, 6)$; $(-3, -6)$; $(-4\sqrt{3}, 5\sqrt{3})$; $(4\sqrt{3}, -5\sqrt{3})$.
19. $(10, -5)$; $(-10, 5)$; $(\frac{5}{2}\sqrt{2}, \frac{15}{2}\sqrt{2})$; $(-\frac{5}{2}\sqrt{2}, -\frac{15}{2}\sqrt{2})$.
21. $[\frac{1}{2}(a + 1), \frac{1}{2}(a - 1)]$; $[\frac{1}{2}(a - 1), \frac{1}{2}(a + 1)]$.
23. 12′ by 5′.
25. $\frac{3}{2}$; $\frac{3}{7}$.
27. 81.
29. 3 lb.
31. $6\frac{3}{5}$ hr.; $5\frac{1}{2}$ hr.

Exercise 104. Page 282

1. 63 (exact).
3. 523 (exact).
5. 325 (exact).
7. 6.39 (exact).
9. 8.85.
11. 40.54.

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